TUTORIAL V:

Continuation of homoclinic orbits with MATCONT

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This session is devoted to location and continuation of orbits homoclinic to hyperbolic equilibria in autonomous systems of ODEs depending on two parameters

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^2,$$

and to detection of their codim 2 bifurcations.

1 Traveling pulses in the FitzHugh-Nagumo model

The following system of partial differential equations is the FitzHugh-Nagumo model of the nerve impulse propagation along an axon:

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \dfrac{\partial^2 u}{\partial x^2} - f_a(u) - v, \\ \dfrac{\partial v}{\partial t} &= bu, \end{cases} \tag{1}$$

where $u = u(x, t)$ represents the membrane potential; $v = v(x, t)$ is a phenomenological "recovery" variable; $f_a(u) = u(u - a)(u - 1),\ 1 > a > 0,\ b > 0,\ -\infty < x < +\infty,\ t > 0$.

Traveling waves are solutions to these equations of the form

$$u(x, t) = U(\xi),\ v(x, t) = V(\xi),\ \xi = x + ct,$$

where c is an a priori unknown wave propagation speed. The functions $U(\xi)$ and $V(\xi)$ satisfy the system of three ordinary differential equations

$$\begin{cases} \dot{U} &= W, \\ \dot{W} &= cW + f_a(U) + V, \\ \dot{V} &= \dfrac{b}{c}U, \end{cases} \tag{2}$$

where the dot means differentiation with respect to "time" ξ. System (2) is called a *wave system.* It depends on three positive parameters (a, b, c). Any bounded orbit of (2) corresponds to a traveling wave solution of the FitzHugh-Nagumo system (1) at parameter values (a, b) propagating with velocity c.

For all $c > 0$ the wave system has a unique equilibrium $0 = (0, 0, 0)$ with one positive eigenvalue λ_1 and two eigenvalues $\lambda_{2,3}$ with negative real parts. The equilibrium can be either a saddle or a saddle-focus and has in both cases a one-dimensional unstable and a two-dimensional stable invariant manifolds $W^{u,s}(0)$. The transition between saddle and saddle-focus cases is caused by the presence of a double negative eigenvalue; for fixed $b > 0$ this happens on the curve

$$D_b = \{(a, c) : c^4(4b - a^2) + 2ac^2(9b - 2a^2) + 27b^2 = 0\}.$$

A branch $W_1^u(0)$ of the unstable manifold leaving the origin into the positive octant can return back to the equilibrium, forming a homoclinic orbit Γ_0 at some parameter values.

For $b > 0$, these parameter values form a curve $P_b^{(1)}$ in the (a, c)-plane that can only be found numerically. As we shall see, this curve passes through the saddle-focus region delimited by D_b. Any homoclinic orbit defines a traveling *impulse*. The shape of the impulse depends on the type of the corresponding equilibrium: It has a monotone "tail" in the saddle case and an oscillating "tail" in the saddle-focus case.

The saddle quantity $\sigma_0 = \lambda_1 + \text{Re}\ \lambda_{2,3}$ is always positive for $c > 0$. Therefore, the phase portraits of (2) near the homoclinic curve $P_b^{(1)}$ are described by Shilnikov's Theorems. In particular, near the homoclinic bifurcation curve $P_b^{(1)}$ in the saddle-focus region, system (2) has an infinite number of saddle cycles. These cycles correspond to *periodic wave trains* in the FitzHugh-Nagumo model (1). Secondary homoclinic orbits existing in (2) near the primary saddle-focus homoclinic bifurcation correspond to *double traveling impulses* in (1). An infinite number of the corresponding secondary homoclinic bifurcation curves $P_{b,j}^{(2)}$ in (2) originate at each point $A_{1,2}$, where $P_b^{(1)}$ intersects D_b.

We will locate a critical value of c for $a = 0.15$ and $b = 0.0025$, at which (2) has a homoclinic orbit, continue this homoclinic orbit with respect to the parameters (a, c), and detect the codim 2 bifurcations points $A_{1,2}$ in $P_b^{(1)}$.

2 System specification

Start a version of MATCONT that supports homoclinic continuation, and specify a new ODE system with the coordinates (`U,W,V`) and time `t`[1]:

```
U'=W
W'=cc*W+U*(U-aa)*(U-1.0)+V
V'=bb*U/cc
```

The parameters a, b, c are denoted by `aa,bb,cc`, respectively. Generate the derivatives of order 1,2, and 3 symbolically.

3 Location of a homoclinic orbit by homotopy

This consists of several steps, each presented in a separate subsection.

3.1 Approximating the unstable manifold by integration

Select **Type**|**Initial point**|**Equilibrium** and **Type**|**Curve**|**ConnectionSaddle**.

In the appearing **Integrator** window, increase the integration `Interval` to 20 (see the right panel of Figure 1).

Via the **Starter** window, input the initial values of the system parameters

[1]Due to MATLAB restrictions, the name `xi` cannot be used here !

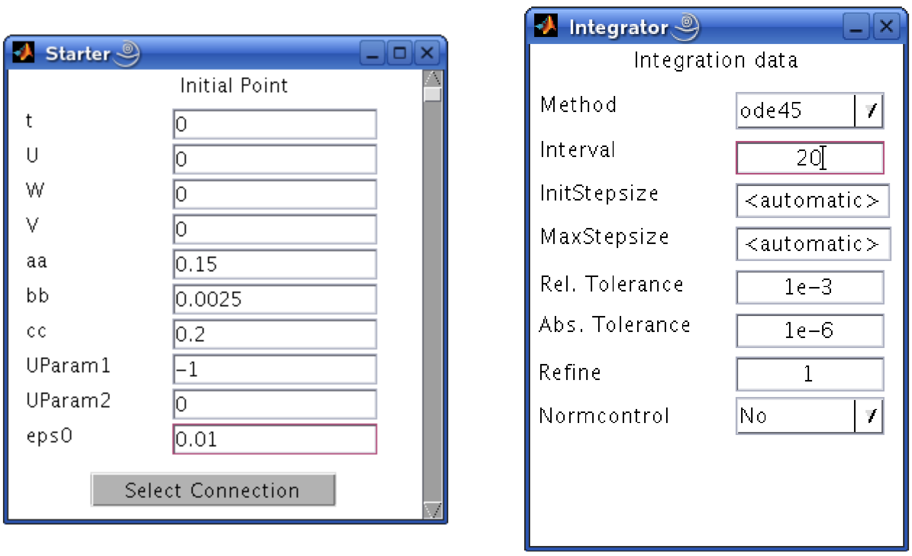

Figure 1: **Starter** and **Integrartor** windows for the integration of the unstable manifold.

```
aa              0.15
bb              0.0025
cc              0.2
```

as well as

```
Uparam1        -1
eps0            0.01
```

that specify direction and distance of the displacement from the saddle

```
x0_U            0
x0_W            0
x0_V            0
```

along the unstable eigenvector[2]. The **Starter** window should look like in left panel of Figure 1.

Open a **2Dplot** window with **Window**|**Graphic**|**2Dplot**. Select `U` and `V` as variables along the corresponding axes and input the following plotting region

```
Abscissa:       -0.2        0.5
Ordinate:       -0.05       0.1
```

Start **Compute**|**Forward**. You will get an orbit approximating the unstable manifold that departs from the saddle in a nonmonotone way, see Figure 2. This orbit does not resemble a

[2]`Uparam2` is only used when $\dim W^u = 2$. On some platforms, `Uparam1`=1 should be used to select the correct direction.

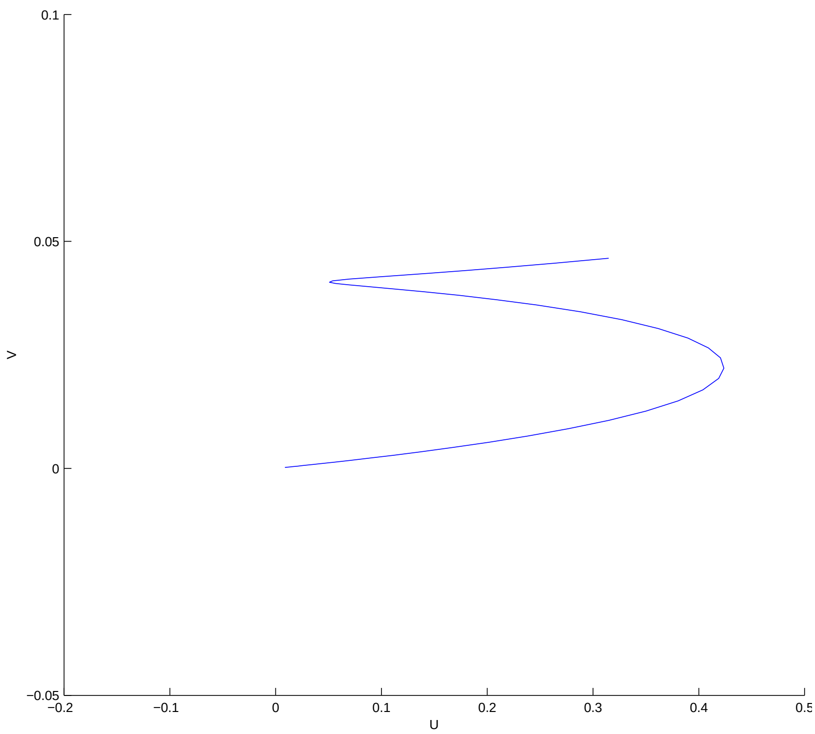

Figure 2: A segment of the unstable manifold of the saddle at the initial parameter values.

homoclinic orbit.

Press `Select Connection` button in the **Starter** window. MATCONT will search for a point in the computed orbit where the distance to the *stable* eigenspace of the Jacobian matrix of the saddle is stopped decreasing for the last time. This point is selected as the end-point of the initial connecting orbit (as we shall see, it corresponds to the time-interval T=8.40218. The program will ask to choose the BVP-discretization parameters `ntst` and `ncol` that will be used in all further continuations. Set `ntst` equal to 50 and keep `ncol` equal to 4 (Figure 3). Press `OK`.

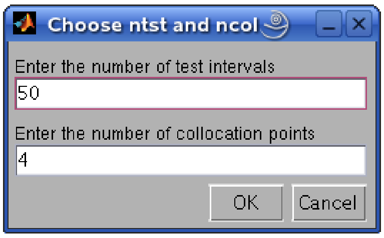

Figure 3: The discretization parameters for homotopy BVPs.

3.2 Homotopy towards the stable eigenspace

In the new **Starter** window, activate the parameters `cc`, `SParam1`, and `eps1` (see Figure 4), and **Compute|Backward**. A family of curves will be produced by continuation (see Figure 5) and the message

```
SParam equal to zero
```

will indicate that the end-point has arrived at the stable eigenspace of the saddle (i.e. reached the plane tangent to the stable invariant manifold at the saddle and given by the condition `SParam1`=0). The corresponding orbit segment is labled HTHom. Stop the continuation there.

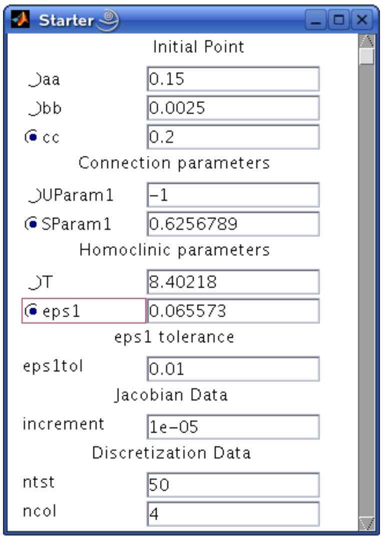

Figure 4: **Starter** window for the homotopy towards the stable eigenspace.

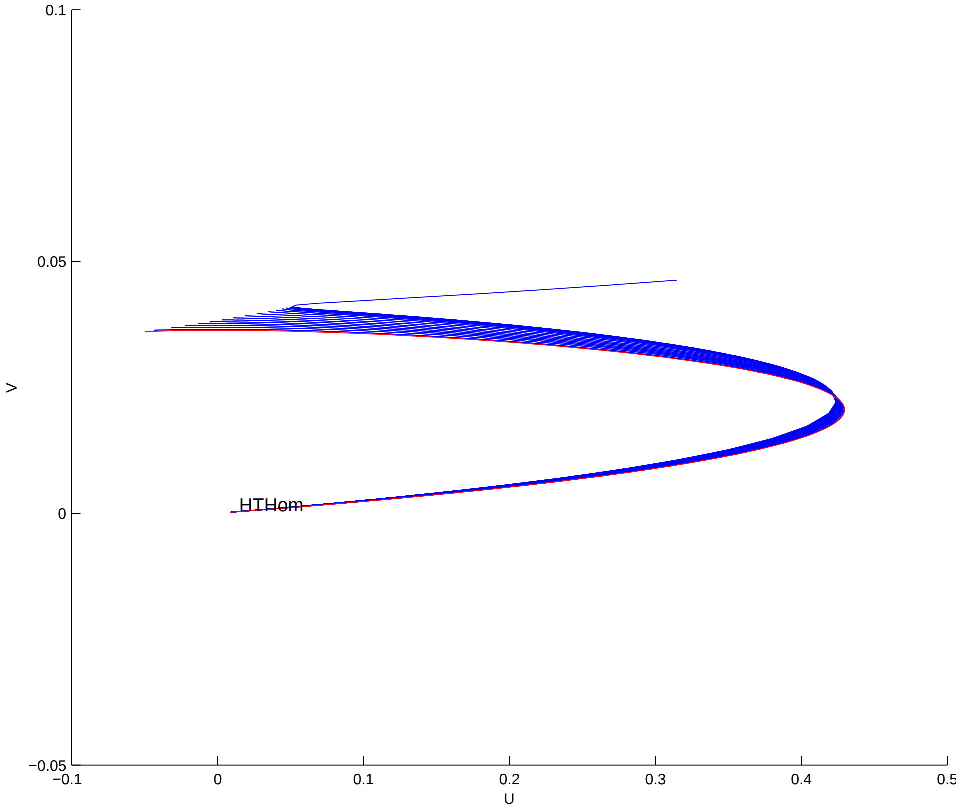

Figure 5: The unstable manifold with the end-point in the stable eigenspace of the saddle.

3.3 Homotopy of the end-point towards the saddle

The obtained segment is still far from the homoclinic orbit but can be selected as the initial point for a homotopy of the end-point towards the saddle. Select

```
2) HTHom: SParam equal to zero
```

via **Select**|**Initial point** menu.

In the **Continuer** window, set `MaxStepsize` to 0.5, see in the right panel of Figure 6.

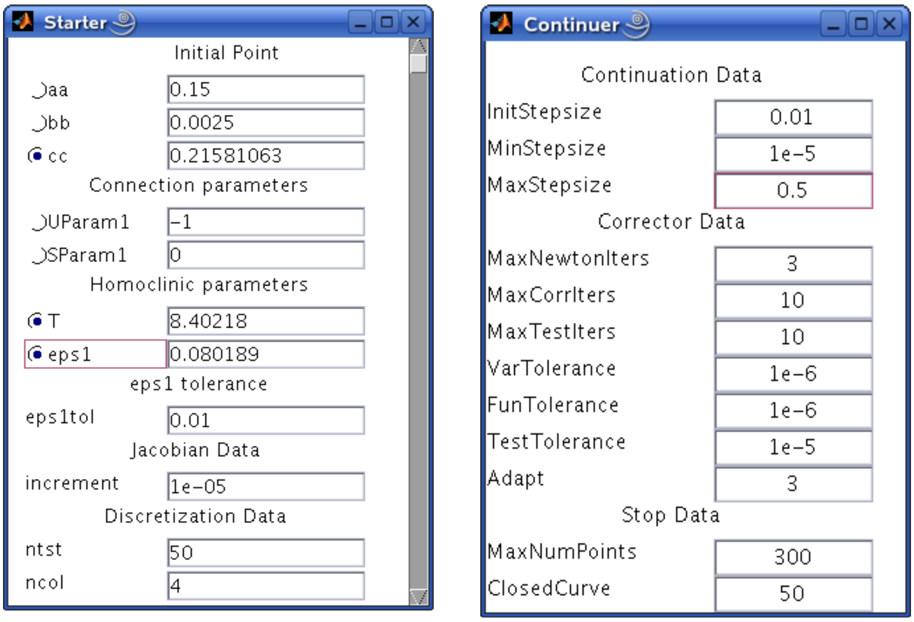

Figure 6: **Starter** and **Integrartor** windows for the homotopy towards the saddle.

In the **Starter** window, `SParam1` now equals to zero, while the parameter `cc` is adjusted. Activate parameters `cc`, `T`, and `eps1` there. Set `eps1tol` equal to 0.01; this will be used as the target distance `eps1` from the end-point to the saddle.

Open a **Numeric** window to monitor the values of the active parameters. Clean the **2DPlot** window and **Compute** | **Forward**. You should get Figure 7, where the last computed segment is again labled by `HTHom`. The message

```
eps1 small enough
```

appears in the main window and indicates that a good approximation of the homoclinic orbit is found. The begin- and the end-points are now both located near the saddle (at distance 0.01). The **Numeric** window at the last computed point is presented in Figure 8. It can be seen that the `eps1` became 0.01, while the time-interval `T` increased to 36.6206. Stop the continuation.

3.4 Continuation of the homoclinic orbit

Select just computed

```
2) HTHom: eps1 small enough
```

via **Select**|**Initial point** menu as the initial data. Select **Type**|**Curve**|**Homoclinic to Saddle** and check that the curve type is `Hom`, while the initial point is of type `HTHom`.

In the new **Starter** window, activate two system parameters: `aa` and `cc` as well as the homoclinic parameter `T` (see Figure 9). These parameters will vary along the homoclinic curve, while

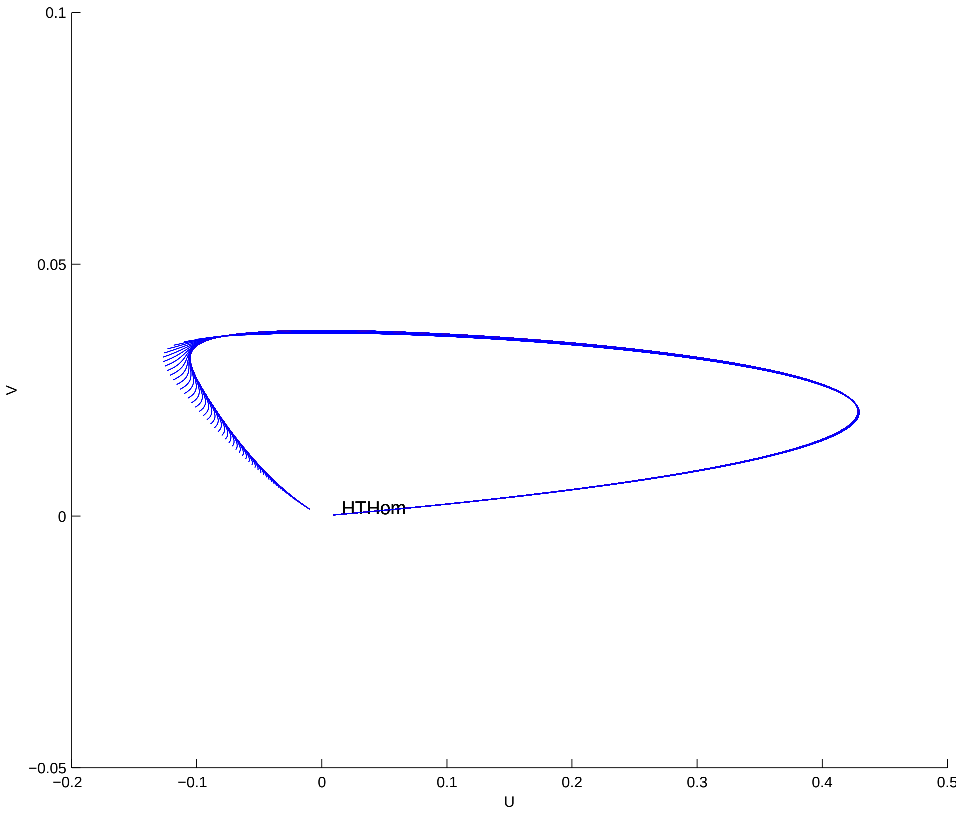

Figure 7: The homotopy results in the manifold segment with both the begin- and the end-points near the saddle.

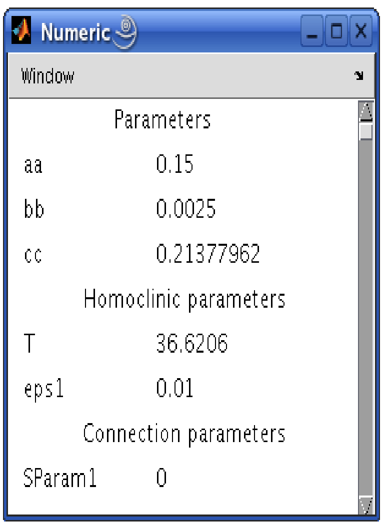

Figure 8: **Numeric** window at the last point of the homotopy towards the saddle.

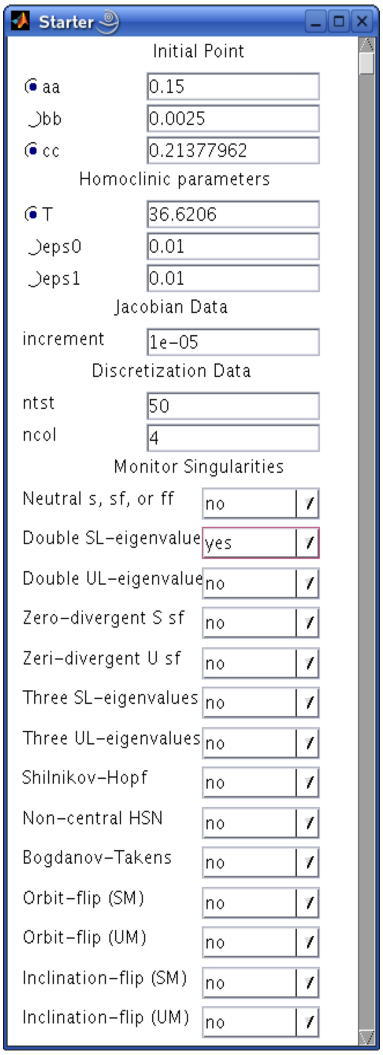

Figure 9: **Starter** window for the two-parameter homoclinic continuation.

both `eps0` and `eps1` (the begin- and end-distances to the saddle) will be fixed, see Figure 9. Also, choose `Yes` to detect the singularity `Double SL-eigenvalue` (*double stable leading eigenvalue*) along the homoclinic curve.

In the **Continuer** window, increase the `MaxStepsize` to 1.

Change the attributes of the **2Dplot** window: Select `aa` and `cc` as the abscissa and ordinate with the visibility limits

```
Abscissa:          0           0.3
Ordinate:          0           0.8
```

Now you are ready to start the continuation. **Compute**|**Forward** and **Backward**, resume computations at special points, and terminate them when the computed points leave the positive quadrant of the (a, c)-plane. Two special points will be detected, where the equilibrium undergoes the saddle-to-saddle-focus transition. These are codim 2 bifurcation points $A_{1,2}$ introduced in Section 1.

Delete all previously computed curves except the last two, namely

```
HTHom_Hom(1)
HTHom_Hom(1)
```

and **Plot**|**Redraw diagram**. This should produce Figure 10.

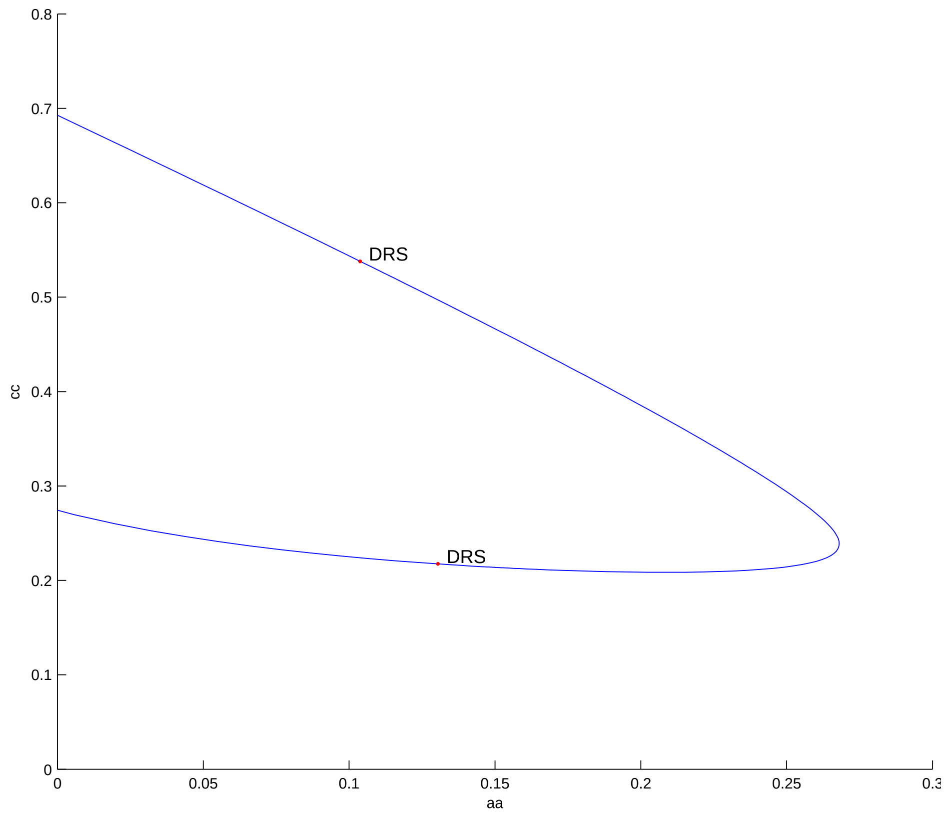

Figure 10: The homoclinic bifurcation curve in the (a, c)-plane. The saddle to saddle-focus transitions $A_{1,2}$ are labled by `DRS`.

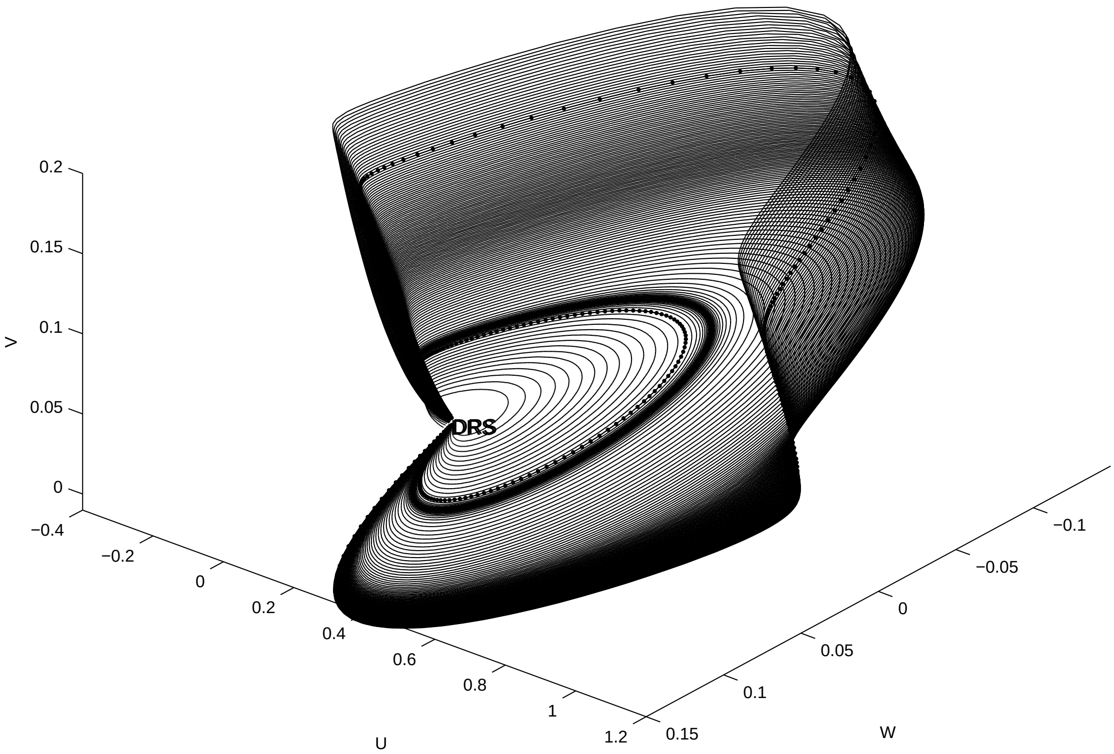

Figure 11: The family of homoclinic homoclinic orbits in the phase space of system (2) for $b = 0.0025$.

To verify that that all computed points indeed correspond to homoclinic orbits, open a **3Dplot** window and select `U,W` and `V` as variables along the coordinate axes with the visibility limits

```
Abscissa:          -0.4      1.2
Ordinate:          -0.15     0.15
Applicate:         -0.01     0.2
```

respectively. **Plot** | **Redraw diagram** in this new window should produce Figure 11 after an appropriate rotation.

4 Additional Problems

A. Consider the famous *Lorenz system*

$$\begin{cases} \dot{x} &=& \sigma(-x+y), \\ \dot{y} &=& rx-y-xz, \\ \dot{z} &=& -bz+xy, \end{cases}$$

with the standard parameter value $b=\frac{8}{3}$. Use MATCONT to analyse its homoclinic bifurcations:

1. Locate at $\sigma=10$ the bifurcation parameter value r_{Hom} corresponding to the primary orbit homoclinic to the origin. *Hint*: Use homotopy starting from $r=15.5$.
2. Compute the primary homoclinic bifurcation curve in the (r,σ)-plane for $b=\frac{8}{3}$. Try to reach $r=100$ and $\sigma=100$.
3. Locate and continue in the same (r,σ)-plane several secondary homoclinic to the origin orbits in the Lorenz system. *Hint*: These orbits make turns around both nontrivial equilibria. The simplest one can be found starting from $(\sigma,r)=(10,55)$.

B. Study with MATCONT homoclinic bifurcations in the *adaptive control system of Lur'e type*:

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& z, \\ \dot{z} &=& -\alpha z-\beta y-x+x^2, \end{cases}$$

where α and β are parameters.