

TUTORIAL II:
One-parameter bifurcation analysis of equilibria with
MATCONT

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This session is devoted to the numerical continuation of equilibria in systems of autonomous ODEs depending on one parameter

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

and detection of their bifurcations.

1 An ecological model with multiple equilibria and limit points

Consider the following system appearing in mathematical ecology:

$$\begin{cases} \dot{x} &= rx(1-x) - \frac{xy}{x+a}, \\ \dot{y} &= -cy + \frac{xy}{x+a} - \frac{dy^2}{y^2+b^2}. \end{cases} \quad (1)$$

Fix

$$r = 2, \quad a = 0.6, \quad b = c = 0.25,$$

and consider d as a bifurcation parameter with initial value $d = 0.1$. The aim is to locate equilibria of (1) and study their dependence on d .

1.1 System specification

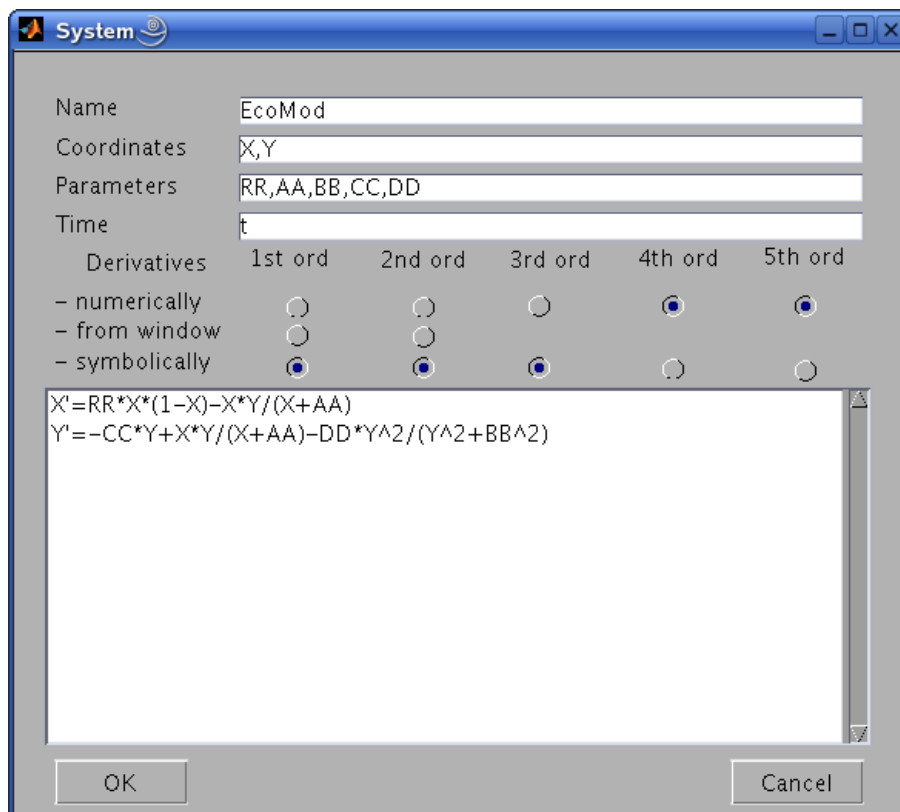


Figure 1: The ecological model.

Start MATCONT, choose **Select|System|New**, and input a new ODE system – say **EcoMod** – into MATCONT as shown in Figure 1.

1.2 Locate an equilibrium by numerical integration

Select **Type|Initial point|Point** and check that the default **Type|Curve|Orbit** is selected.

In the appearing **Integrator** window, increase the integration **Interval** to 200 and set **Rel.Tolerance** to $1e-7$ and **Abs.Tolerance** to $1e-10$ (to assure smaller integration steps).

Via the **Starter** window, input the initial point

```
X      1.2
Y      1
```

and the initial values of the parameters, namely

```
RR      2
AA      0.6
BB      0.25
CC      0.25
DD      0.1
```

Open a **2Dplot** window with **Window|Graphic|2Dplot**. Select **X** and **Y** as variables along the corresponding axes and input – after clicking **Layout|Plotting region** – the following visibility limits

```
Abscissa:    0      1.2
Ordinate:    0      2.0
```

Start **Compute|Forward**. You will get an orbit converging to an equilibrium, see Figure 2

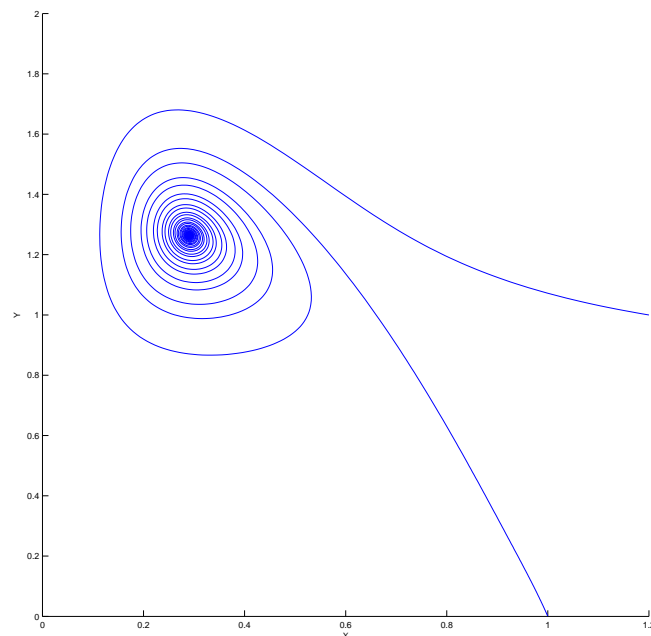


Figure 2: Phase orbits approaching a stable equilibrium for $d = 0.1$

To see how the coordinates vary along the orbit, select **Window|Numeric** which opens a **Numeric** window. Then input another initial point in the **Starter** window

```
X      1.0
Y      0.001
```

and **Compute|Forward**. You get the second orbit from Figure 2 tending to the same equilibrium, whose (approximate) coordinates can be seen in the **Numeric** window shown in Figure 3. Close

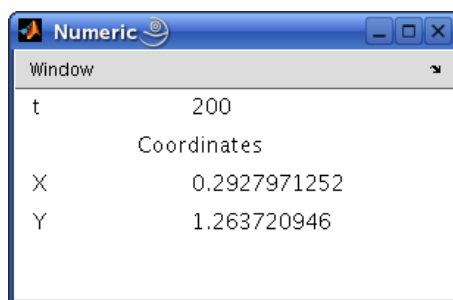


Figure 3: The coordinates of the stable equilibrium.

the **Numeric** window.

1.3 Equilibrium curve

We can continue the equilibrium found by integration with respect to the parameter d (i.e. **DD**).

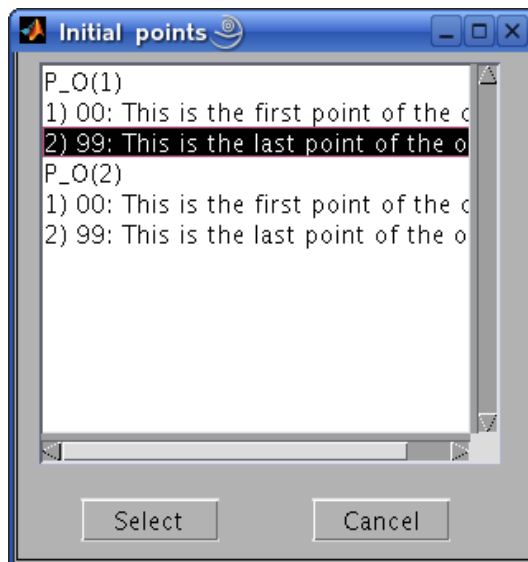


Figure 4: **Initial points** window.

For this, open the **Initial points** window with **Select|Initial point** and choose the last point of the orbit **P_O(1)** (click to highlight and then press **Select** button, see Figure 4). To tell **MATCONT** to continue an equilibrium curve, we have to specify the initial point type and the curve type. Selecting **Type|Initial point|Equilibrium** we set the point type **EP** and the same (default) curve type, so that we prepare to compute the **EP_EP** equilibrium curve, as indicated in the main **MatCont** window. Two new windows appear: **Starter** and **Continuer**.

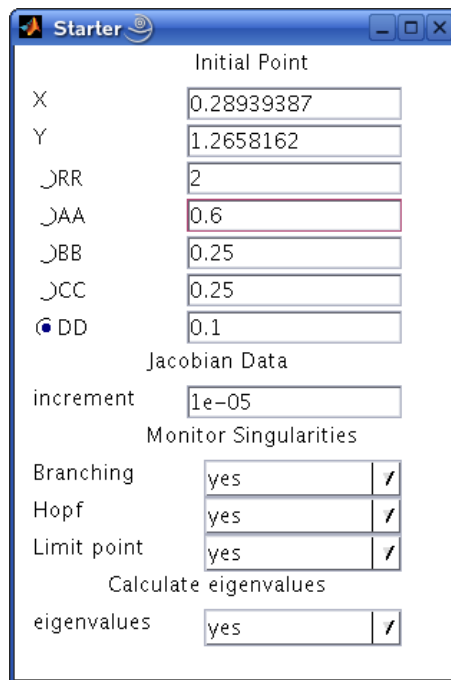


Figure 5: **Starter** window for the equilibrium continuation: Parameter DD is activated.

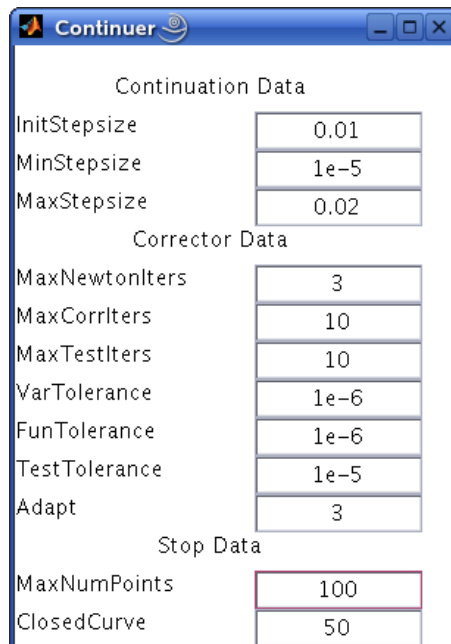


Figure 6: Continuation parameters in the **Continuer** window.

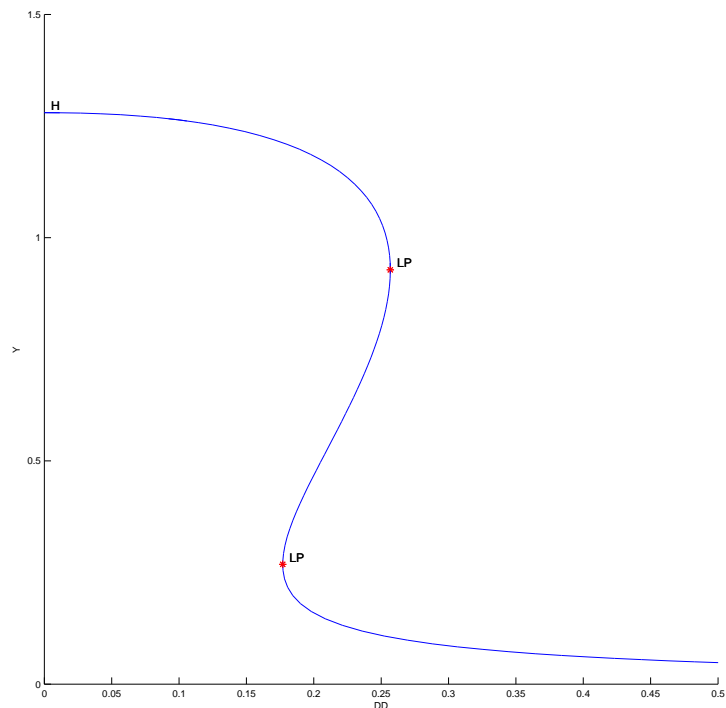


Figure 7: Equilibrium manifold in the (d, y) -plane: LP's denote limit points, H is a Hopf point.

The initial values of the parameters and the equilibrium coordinates are visible in the **Starter** window. Press the radio-button next to parameter DD, indicating that it will be a bifurcation parameter (i.e. *activate* parameter d). The resulting **Starter** window is shown in Figure 5. There you can also see which singularities will be monitored, and whether the equilibrium eigenvalues will be computed.

In the **Continuer** window, several default numerical parameters related to the continuation are listed. Change only

```
MaxStepsize      0.02
MaxNumPoints     100
```

(see Figure 6).

Use **Layout** menu in the **2Dplot** window to select the parameter DD as abscissa and the coordinate Y along as ordinate with the visibility limits

```
Abscissa:        0          0.5
Ordinate:        0          1.5
```

Now **Compute|Forward** and **Compute|Backward** to get a sigmoidal curve as in Figure 7. Along the forward branch, MATCONT stops at two *limit points* labeled LP. You can **Resume** the computation at these points in the small control window. Along the backward branch, MATCONT stops at a *Hopf point* labeled H. Terminate the computation by pressing **Stop** there.

To determine stability of the equilibria and read the bifurcation parameter values, open **Window|Numeric** and select **Window|Layout** in the **Numeric** window. Click **eigenvalues** turning them into EIGENVALUES, see Figure 8. This makes the eigenvalues of the equilibrium visible in the **Numeric** window (upon resizing it).

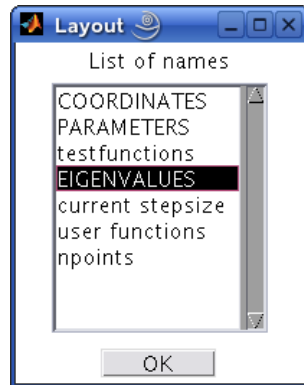


Figure 8: Show eigenvalues in the **Numeric** window.

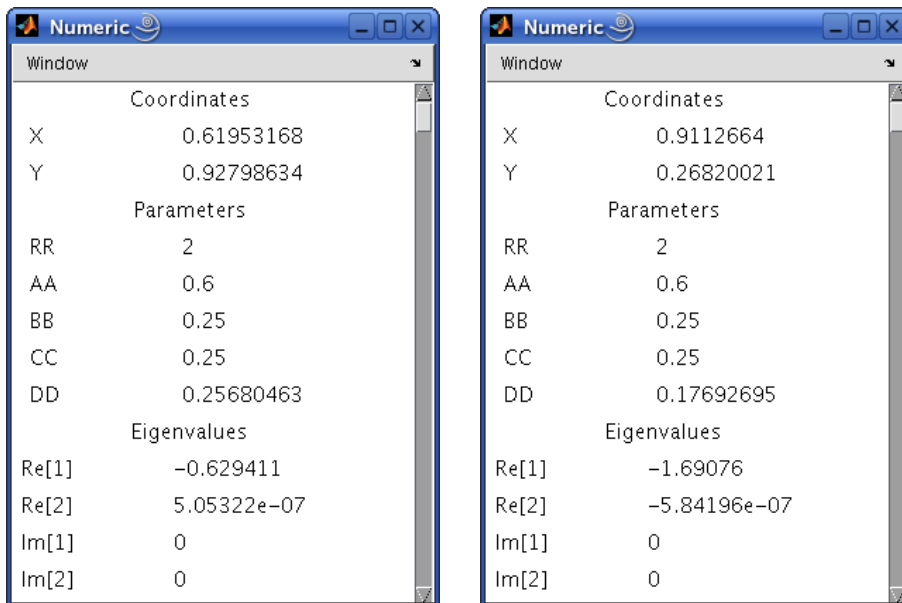


Figure 9: **Numeric** windows at the first and second limit point.

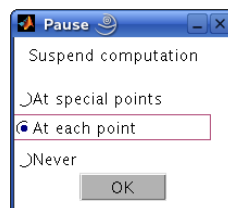


Figure 10: Pause at each computed point.

Clear the **2Dplot** window with **Plot|Clear** and repeat **Compute|Forward**. In the **Numeric** window you can read the LP parameters:

$$d_2 = 0.256805\dots, \quad d_1 = 0.176927\dots$$

(see Figure 9). One eigenvalue (close to) zero is also present in each case. The existence of three equilibria for $d \in (d_1, d_2)$ is evident from the figure.

In the MATLAB Command Window, the value of the *fold normal form* coefficient a is shown at each limit point:

```
label = LP, x = ( 0.619532 0.927986 0.256805 )
a=-5.311546e-01
label = LP, x = ( 0.911266 0.268200 0.176927 )
a=5.681504e-01
```

Finally, clear the **2Dplot** window once more and recompute the equilibrium curve forward with **Options|Pause** mode selected as shown in Figure 10. Resume computations after each computed point and monitor the eigenvalues in the **Numeric** window. You should see that, indeed, the upper and lower branches correspond to linearly stable equilibria (with all $\text{Re } \lambda_i < 0$), while the middle branch correspond to a linear saddle (with $\lambda_1 < 0 < \lambda_2$).

Since $a \neq 0$ to within numerical accuracy, the equilibrium curve is approximately a (scaled) parabola near each of them.

Restore the original pause option (**At special points**) via **Option|Pause** and close the **Numeric** window.

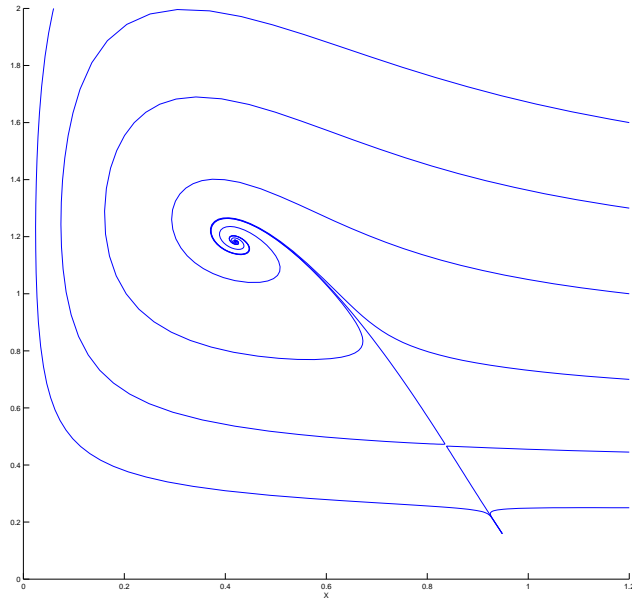


Figure 11: Two stable equilibria (a focus and a node) and a saddle in the ecological model at $d = 0.2$. The stable manifold of the saddle separates the domain of attraction of the focus from that of the node.

1.4 Phase portrait

To verify our conclusions, compute several orbits to get a phase portrait of the system at $d = 0.2$ as in Figure 11. Start always at $x = 1.2$ but with different y 's.

2 Limit and branching points in a discretization of Bratu-Gelfand PDE

Consider the following evolution problem for $u = u(x, t)$ with $x \in [0, 1], t \geq 0$:

$$u_t = u_{xx} + \lambda e^u, \quad u(0) = u(1) = 0, \quad (2)$$

where λ is a parameter. It is called the *Bratu-Gelfand* problem. In this section we study the behaviour of the stationary solution to (2) as function of λ using a finite-difference approximation over an equidistant mesh. One such time-independent solution is obvious: $u \equiv 0$ for $\lambda = 0$.

2.1 Discretization

Characterize a solution to (2) at time t by its values $u_i(t) = u(x_i, t)$ at the uniformly distributed mesh points

$$x_i = ih, \quad h = \frac{1}{N+1}, \quad i = 0, 1, \dots, N, N+1,$$

for some $N > 1$. Then approximate the spatial derivative in (2) by finite differences at the inner points

$$\dot{u}_i(t) = \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2} + \lambda \exp(u_i) = 0, \quad i = 0, 1, 2, \dots, N, \quad (3)$$

and add the boundary conditions in the form

$$\begin{cases} u_0(t) = 0, \\ u_{N+1}(t) = 0. \end{cases} \quad (4)$$

After elimination $u_0(t)$ and $u_{N+1}(t)$, the equations (3) and (4) take the form of the ODE-system

$$\dot{U} = F(U, \lambda), \quad (5)$$

where $U = (u_1, u_2, \dots, u_N)$ and $F : \mathbb{R}^{N+1} \times \mathbb{R} \rightarrow \mathbb{R}^{N+1}$ is given by (3). An equilibrium of this system approximates a stationary solution to (2) with the $O(h^2)$ -accuracy.

In this section, we will study a crude spatial discretization of (2) with only two internal points (i.e. $N = 2$ and $h = \frac{1}{3}$), so that (5) becomes

$$\begin{cases} \dot{u}_1 = -2u_1 + u_2 + \alpha e^{u_1}, \\ \dot{u}_2 = u_1 - 2u_2 + \alpha e^{u_2}, \end{cases} \quad (6)$$

where $\alpha = h^2\lambda = \frac{1}{9}\lambda$ and time is also scaled. The system (6) has a \mathbb{Z}_2 -symmetry. It is invariant under the involution $(u_1, u_2) \mapsto (u_2, u_1)$.

We will compute the equilibrium manifold of (6) in the (u_1, u_2, α) -space starting from point $(0, 0, 0)$ corresponding to the trivial stationary solution of (2), namely $u(x, t) \equiv 0$.

2.2 System specification

Input a new ODE system into MATCONT, namely

```
u1'=-2*u1+u2+alpha*exp(u1)
u2'=u1-2*u2+alpha*exp(u2)
```

and choose to generate the partial derivatives of order 1 and 2 symbolically.

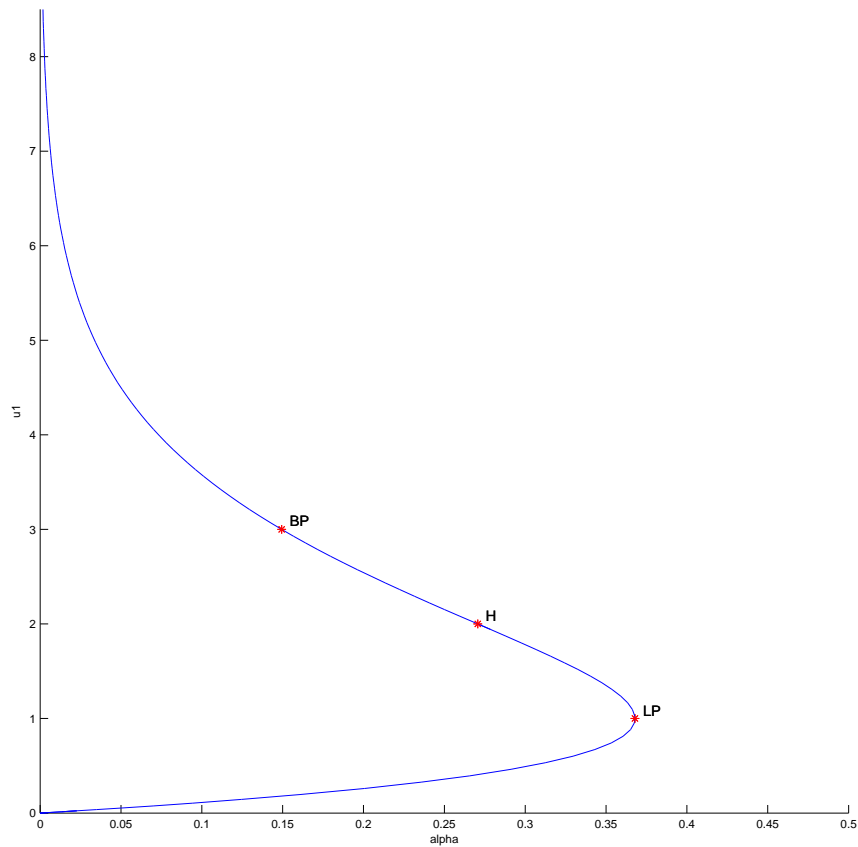


Figure 12: The curve of symmetric equilibria of (6) with limit point LP, neutral saddle H, and branching point BP.

2.3 Symmetric equilibrium branch

Select **Type|Initial point|Equilibrium**. Two windows will appear: **Starter** and **Continuer** windows corresponding to the equilibrium continuation (curve type EP_EP).

In the **Starter** window, activate the parameter **alpha**. Since (6) has an equilibrium $u_1 = u_2 = 0$ for $\alpha = 0$, no further changes to the **Starter** parameters are required.

Open a **2Dplot** window and select the parameter **alpha** and the coordinate **u1** as abscissa and ordinate, respectively, with the visibility limits

```
Abscissa:      0      0.5
Ordinate:     0      8.0
```

To monitor stability of the equilibrium, open a **Numeric** window and select **EIGENVALUES** to be visible in it via **Window|Layout**.

Click **Compute|Forward**. The first bifurcation point, namely a *limit point* LP, will be located at $\alpha = 0.367879\dots$. In MATLAB Command Window, you can read the following message:

```
label = LP, x = ( 1.000001 1.000001 0.367879 )
a=3.535537e-01
```

that gives the critical equilibrium coordinates and the critical parameter value, as well as the quadratic normal form coefficient $a = 0.35355\dots$ at LP. Thus, the limit point is nondegenerate and the equilibrium manifold near LP looks like a parabola.

Further computation detects a *neutral saddle* H at $\alpha = 0.270671\dots$, where the equilibrium has eigenvalues $\lambda_{1,2} = \pm 1$, so that their sum is zero. This is not a bifurcation point for the equilibrium, since it is a hyperbolic saddle.

Resume computations to get the second bifurcation point at $\alpha = 0.14936122\dots$, namely, a *branching point* BP. Extend the equilibrium curve further until you leave the window. You should get Figure 12.

Notice that along the whole computed branch we have $u_1 = u_2$, i.e. it is composed of *symmetric equilibria*. Check by looking at the eigenvalues in the **Numeric** window that only the lower part of the branch is stable, while it is a saddle between LP and BP, and a repeller above it.

Rename the computed curve via **Select|Curve** followed by **Actions|Rename...** to *symmetric*.

2.4 Symmetry breaking

Let us compute the second branch passing through the branching point BP. For this, open with **Select|Initial point** the **Initial points** window and select the BP: **Branch point** of the equilibrium curve *symmetric*. The (u_1, u_2, α) -values corresponding to BP will be loaded.

Select **Type|Curve|Equilibrium**. In the appearing **Starter** window, select **no** for all singularities to monitor as in Figure 13.

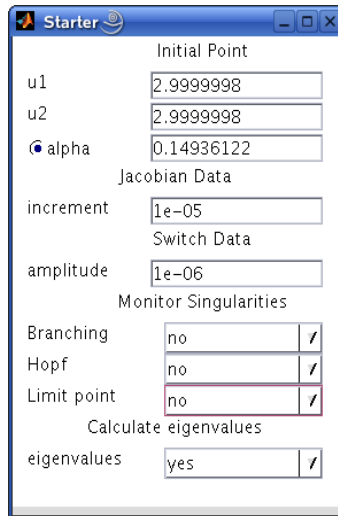


Figure 13: **Starter** window to switch branches at BP.

Change nothing in the **Continuer** window but **Compute|Forward** and **Compute|Backward**. You will get Figure 14. Thus, the branching point BP corresponds, actually, to a *pitchfork bifurcation*: Two more branches of equilibria bifurcate vertically from this point. Moreover, each branch is composed of equilibria which are not symmetric, $u_1 \neq u_2$. However, one branch is mapped into another by the involution. This phenomenon is called the *symmetry breaking*: We got nonsymmetric equilibrium solutions in a symmetric system.

Warning: We have shown numerically that a *discretization* of the Bratu-Gelfand problem (2) has multiple stationary solutions and limit and branching points. It does not imply immediately that the original PDE problem has the same properties. To verify this, one has to repeat the continuation with smaller h (i.e., bigger N) and eventually make some error estimates. It should be noted that the appearance of the pitchfork is due to the equidistant mesh with a small number

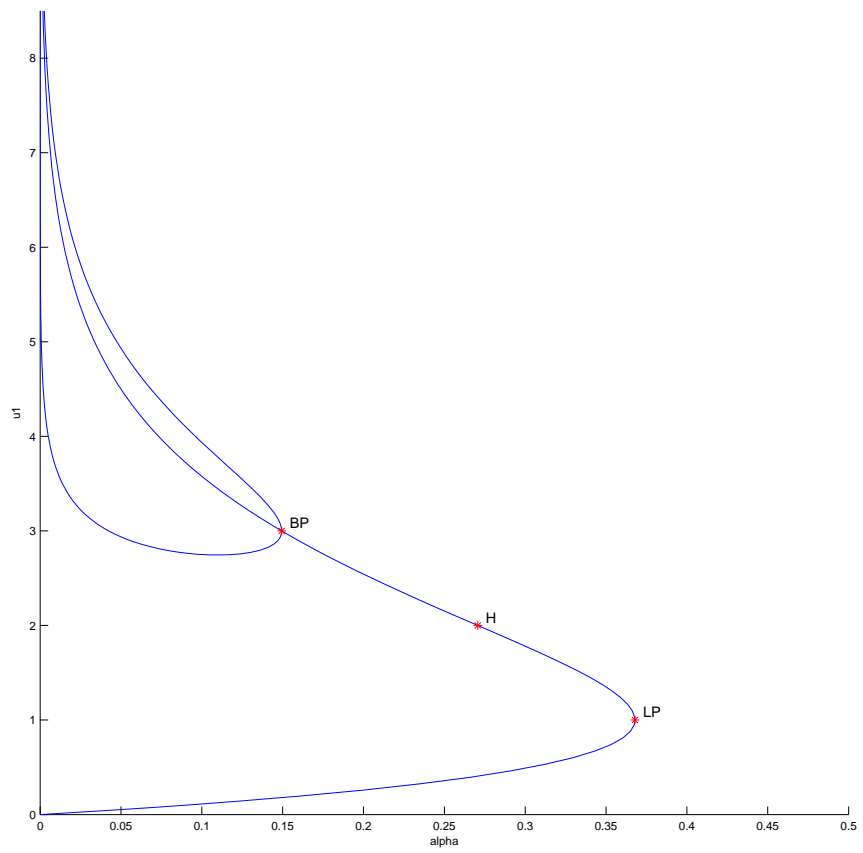


Figure 14: Two equilibrium branches passing through point BP.

of points. If more accurate discretizations of (2) are used, the pitchfork and associated branches disappear. The limit point remains and corresponds to a limit point on the branch of the stationary solutions of (2).

2.5 Stop at zero of a user function

Sometimes we want to stop computation at a prescribed parameter value. This can be done in MATCONT by specifying a *user function*. For example, let us compare two nonsymmetric equilibria at $\alpha = 0.1$.

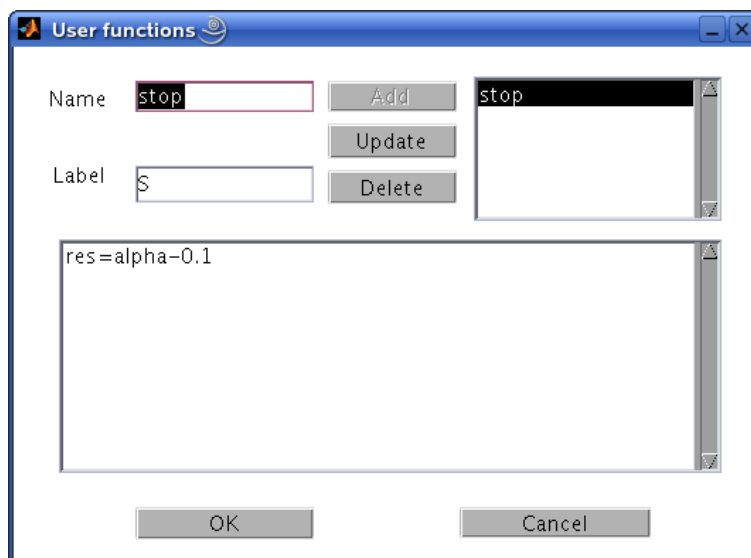


Figure 15: User function input.

Click **Select|User function** in the main **MatCont** window and define a user function **stop** with the associated label **S** by typing `res=alpha-0.1` in the edit field of the **User functions** window (see Figure 15). Next press **Add** and then **OK** buttons. The **Starter** window will change and show a user function control. If one wants to monitor the user function, the **USERFUNCTIONS** must be activated in the layout of the **Numeric** window.

Recomputing the nonsymmetric branches with **Compute|Forward** and **Compute|Backward** will stop at two points labeled **S**, where the user-defined function **stop** vanishes. You can see the corresponding equilibrium coordinates and eigenvalues in the **Numeric** window.

3 Additional Problems

- A. Compute the equilibrium manifold of the scalar ODE

$$\dot{y} = x^2 + y^2 - 1, \quad y \in \mathbb{R}, \quad (7)$$

where $x \in \mathbb{R}$ is a parameter.

- B. Consider the following chemical model

$$\begin{cases} \dot{x} &= 2q_1z^2 - 2q_5x^2 - q_3xy, \\ \dot{y} &= q_2z - q_6y - q_3xy, \\ \dot{s} &= q_4z - Kq_4s, \end{cases}$$

where $z = 1 - x - y - s$ and

$$q_1 = 2.5, \quad q_2 = 1.55, \quad q_3 = 10, \quad q_4 = 0.0675, \quad q_5 = 1.0, \quad q_6 = 0.1,$$

and

$$K = 2.0.$$

Given an equilibrium $(x, y, s) = (0.0032\dots, 0.8838\dots, 0.0376\dots)$, find two more positive equilibria of the system.

Hint: Compute a curve in the (x, y, s) -space defined by the first two equations:

$$\begin{cases} 2q_1z^2 - 2q_5x^2 - q_3xy = 0, \\ q_2z - q_6y - q_3xy = 0, \end{cases}$$

and detect zeros of the user-defined function $F = q_4z - Kq_4s$ along this curve¹.

C. Find a real eigenvalue and the corresponding eigenvector of the matrix

$$B = \begin{pmatrix} -5 & 2 & -3 \\ -5 & 0 & -4 \\ 8 & -2 & 6 \end{pmatrix}$$

by continuation of the eigenvector $v_1 = (0, 0, 1)^T$ of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

corresponding to its eigenvalues $\lambda_1 = -2$.

1. Consider a one-parameter family of matrices

$$C(\alpha) = \alpha B + (1 - \alpha)A, \quad C(0) = A, \quad C(1) = B,$$

and the following continuation problem in the (v, λ, α) -space:

$$\begin{cases} C(\alpha)v - \lambda v = 0, \\ v^T v - 1 = 0. \end{cases}$$

Explain, how it can be used to solve the formulated problem.

2. Setup an auxiliary 4D-system of ODEs

$$\begin{cases} \dot{v} = C(\alpha)v - \lambda v, \\ \dot{\lambda} = v^T v - 1 \end{cases}$$

and continue its equilibrium manifold in the (v, λ, α) -space. Use $(v_1, \lambda_1, 0)$ as the initial point. Stop when $\alpha = 1$ using a user-defined function.

- Continue other two eigenvectors of A and produce a plot tracing all three real eigenvalues in the (α, λ) -plane. What happens in the limit points of these curves?
- Can you setup a continuation problem for (real and imaginary) parts of complex eigenvalues and eigenvectors of $C(\alpha)$?

¹As described in Section 2.5, click **Select** | **User function** in the **MatCont** window and type the expression for F after **res=** in the edit field of the appearing **User functions** window. Fill in the **Name** and **Label** fields, and press **Add** and **OK**.

D. Study equilibria of the following *complex* ODE:

$$\dot{z} = \alpha + z^2, \quad z \in \mathbb{C}^1, \quad (8)$$

where the parameter α and the time are real. *Hint:* Write $z = x + iy$, where (x, y) are real. Then (8) takes the form

$$\begin{cases} \dot{x} &= \alpha + x^2 - y^2, \\ \dot{y} &= 2xy. \end{cases} \quad (9)$$

At $\alpha = -1$ this system has equilibrium $(x_0, y_0) = (1, 0)$, which can be continued.

Plot also the phase portraits in the (x, y) -plane of (9) for $\alpha = -0.25$, $\alpha = 0$, and $\alpha = 0.25$

E. Study discretizations of the Bratu-Gelfand problem in more detail:

1. For $N = 2$, find *analytically* the parameter value α_{BP} corresponding to the branching point in system (6) and compare it with the numerical result in Section 2.
2. Analyse numerically the effect of increasing the number of equidistant mesh points to $N = 3$, $N = 4$, and $N = 10$ on the λ -values corresponding to the limit and branching points in the finite-difference approximation (3)-(4) of the Bratu-Gelfand problem (2).

Hint: Do not forget that $\lambda = (N + 1)^2\alpha$.