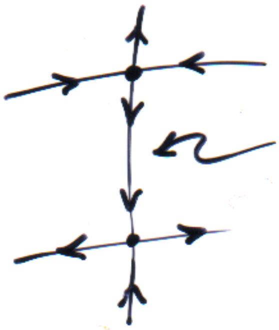
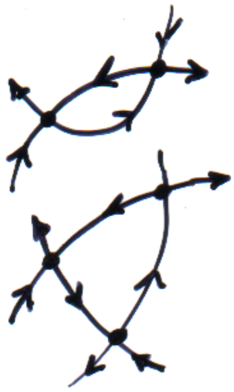
Homoclinic and heteroclinic connections

A trajectory connecting two saddles is called <u>heteroclinic connection</u>

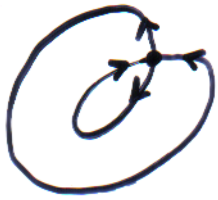


heteroclinic connection

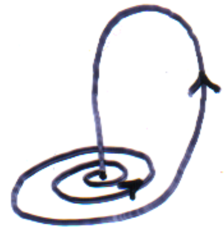


heteroclin. loops

A trajectory connecting a saddle to itself is called <u>homoclinic connection</u> (or homoclinic orbit or homoclinic loop)



double homoclinic loop



homocl. in $\mathbb{R}^3$

(a)

(b)

(c)

The region inside the homoclinic loop is an invariant set

(a) attracting homoclinic loop

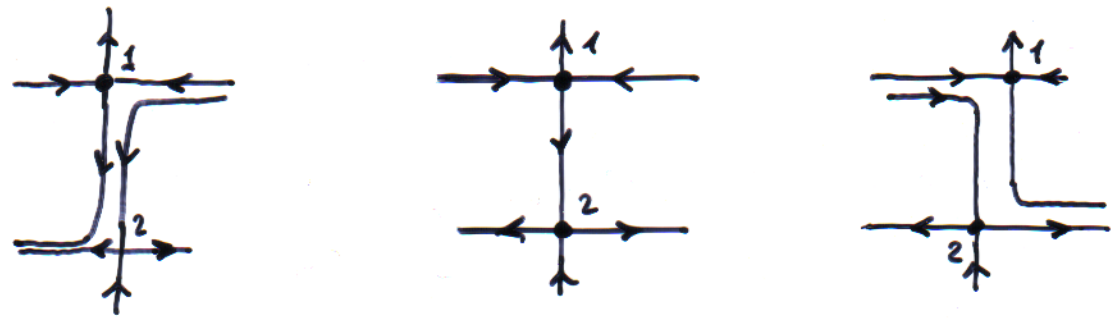
(b) neutral " "

(c) repelling " "

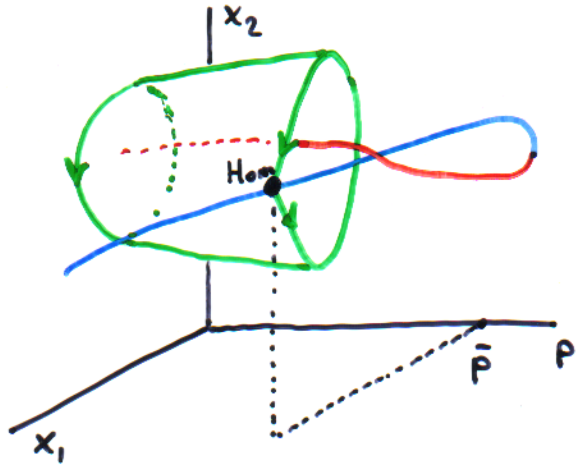
inside the loop in case (a)

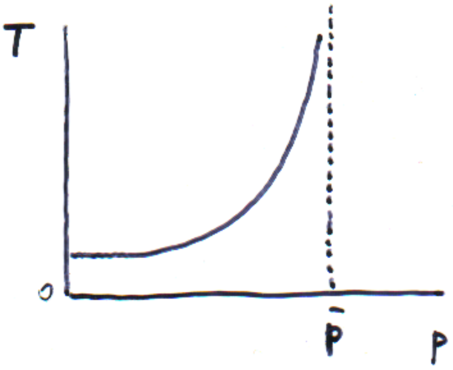
Heteroclinic and homoclinic bifurcations

In general, a system with an heteroclinic or homoclinic connections is structurally unstable.

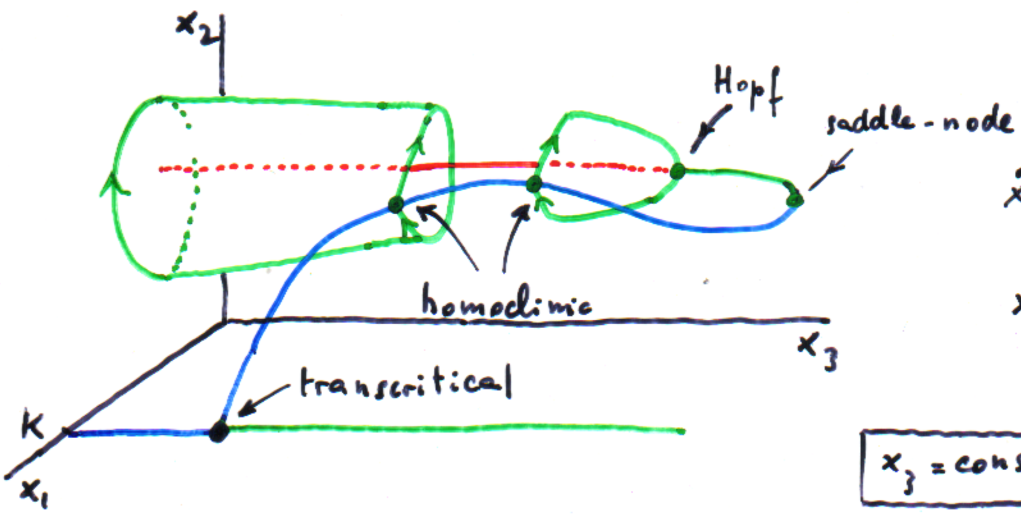


$\bar{p} - \varepsilon$ $\qquad\qquad$ $\bar{p}$ $\qquad\qquad$ $\bar{p} + \varepsilon$





Ex. 1 Tritrophic food chain

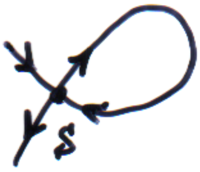


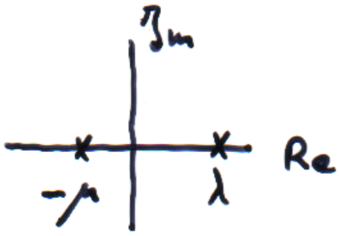
$$\dot{x}_1 = r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{a x_1}{b + x_1} x_2$$

$$\dot{x}_2 = e \frac{a x_1}{b + x_1} x_2 - m x_2 - \frac{c x_2}{d + x_2} x_3$$

$\boxed{x_3 = \text{const.}}$

Andronov's theorem in $\mathbb{R}^2$



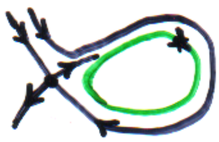


$\sigma = \lambda - \mu$
↑ saddle value

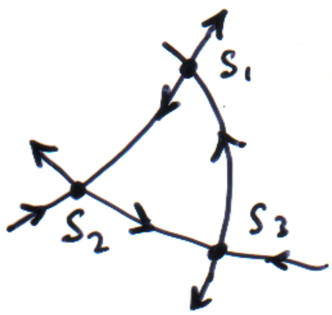
Close to the saddle : $e^{\lambda t}$ against $e^{-\mu t}$

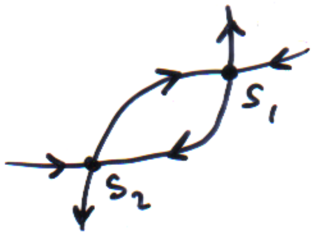
$\sigma < 0 \; [\log \mu > \log \lambda]$
⇓
the homoclinic loop attracts
⇓
the cycle obtained through perturbation is stable

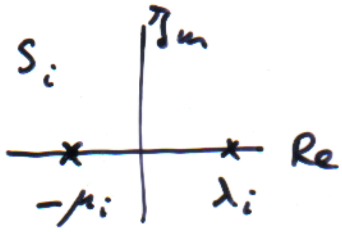
$\sigma > 0 \; [\log \mu < \log \lambda]$
the homoclinic loop is repelling
⇓
the cycle obtained through perturbation is unstable



Extension to heteroclinic loops [Reyn 1979]



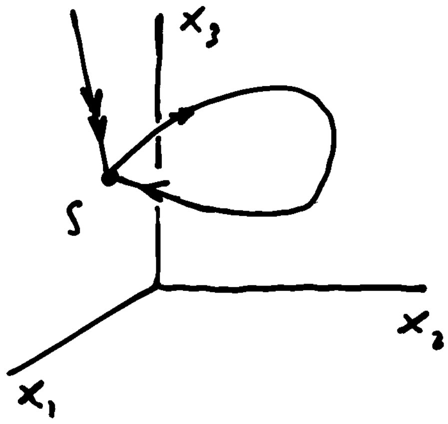


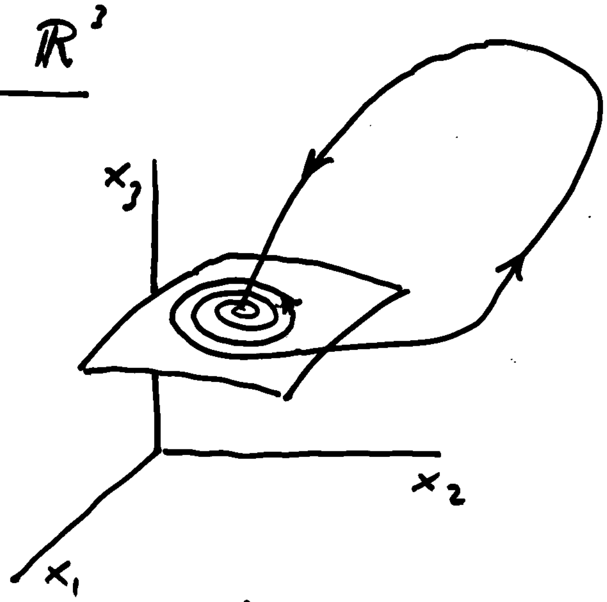


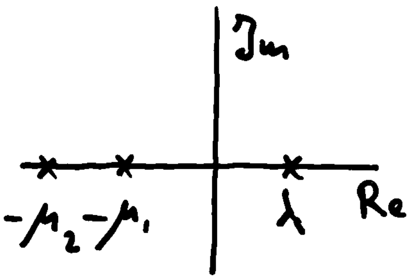
If a parameter is perturbed the loop breaks and a cycle appears (if the perturbation has the right sign)
Moreover

$$\sum \log \mu_i > \sum \log \lambda_i \;\Rightarrow\; \text{stable cycle}$$
$$\sum \log \mu_i < \sum \log \lambda_i \;\Rightarrow\; \text{unstable cycle}$$

Shilnikov's theorem in $\mathbb{R}^3$

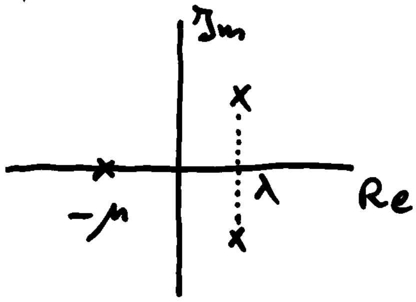






real saddle

$\sigma = \lambda - \mu_1$



complex saddle

$\sigma = \lambda - \mu$

eigenvalue closest to imaginary axis:

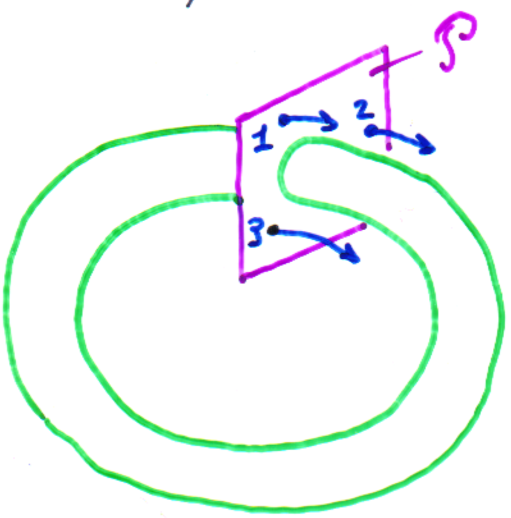
- real
  - positive ($\sigma < 0$) $\Rightarrow$ stable cycle
  - negative ($\sigma > 0$) $\Rightarrow$ unstable cycle
- complex
  - positive real part $\Rightarrow$ ∞ cycles
  - negative real part $\Rightarrow$ ∞ cycles

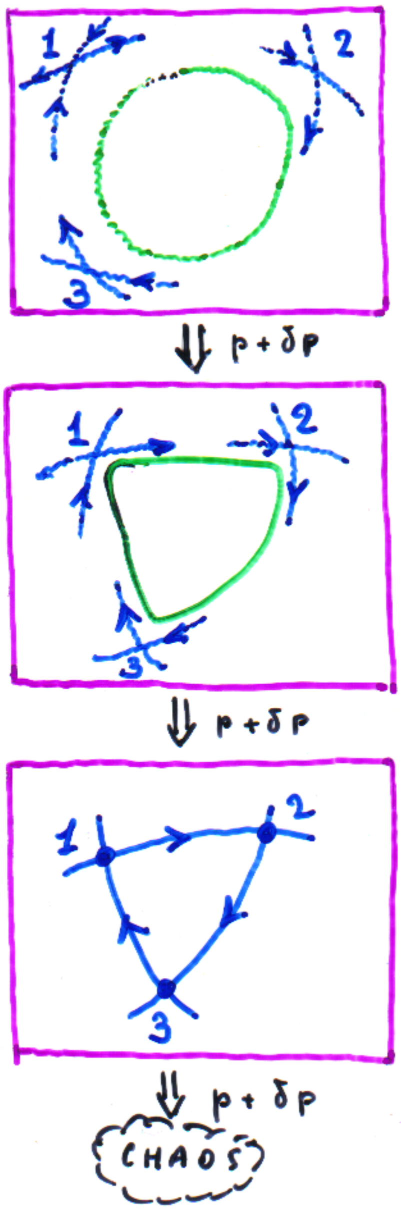
  } "chaos"

- The theorem holds provided the saddle is not degenerate (11 conditions! Champneys-Kuznetsov 1994)
- The theorem has been extended to $\mathbb{R}^n$.

Torus destruction

Torus destruction is a second route to chaos. It is characterized by a single (but global) bifurcation, namely an heteroclinic bifurcation.





torus and period 3 cycle

pinched torus

collision : heteroclinic loop