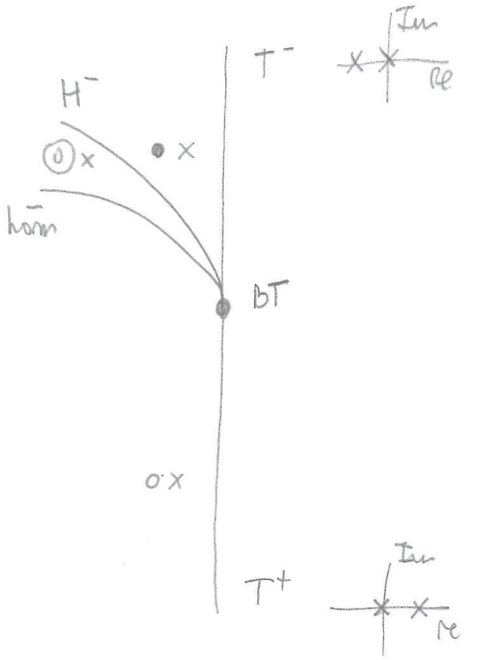
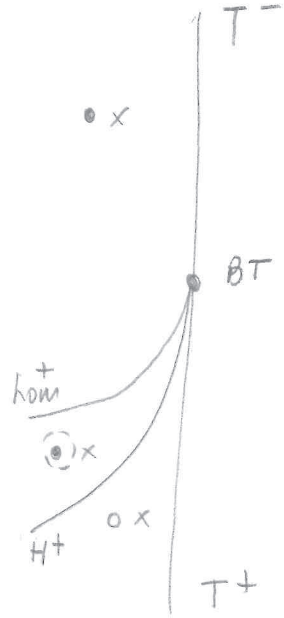
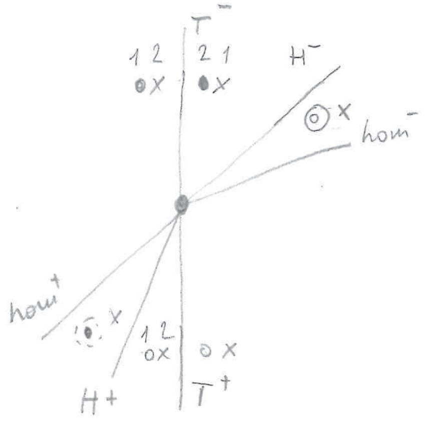
Bogdanov-Takens (double-zero) bifurcation

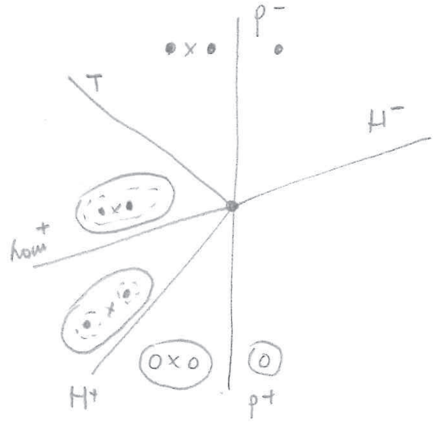
- $n = 2$
- $J(\bar{x}(p^*), p^*) \neq 0$ has two zero eigenvalues (Jordan form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$)
- other genericity conditions to have codim $= 2$
- the discrete-time case is more complex ...
- along a fold / along a Hopf / normal form $\Rightarrow$ two cases





- recall that the node-focus transition is not a bifurcation
- note: $H^{\mp}$ e $hom^{\mp}$ are tangent to $T^{\mp}$ (from the normal form analysis)
  ($hom^{\mp}$ requires $\sigma \lessgtr 0$; $\sigma \to 0$ from below/above at BT along T^-/T^+)
- the case with transcritical (the case with pitchfork is due to Korasov-Takens, see Carr's book





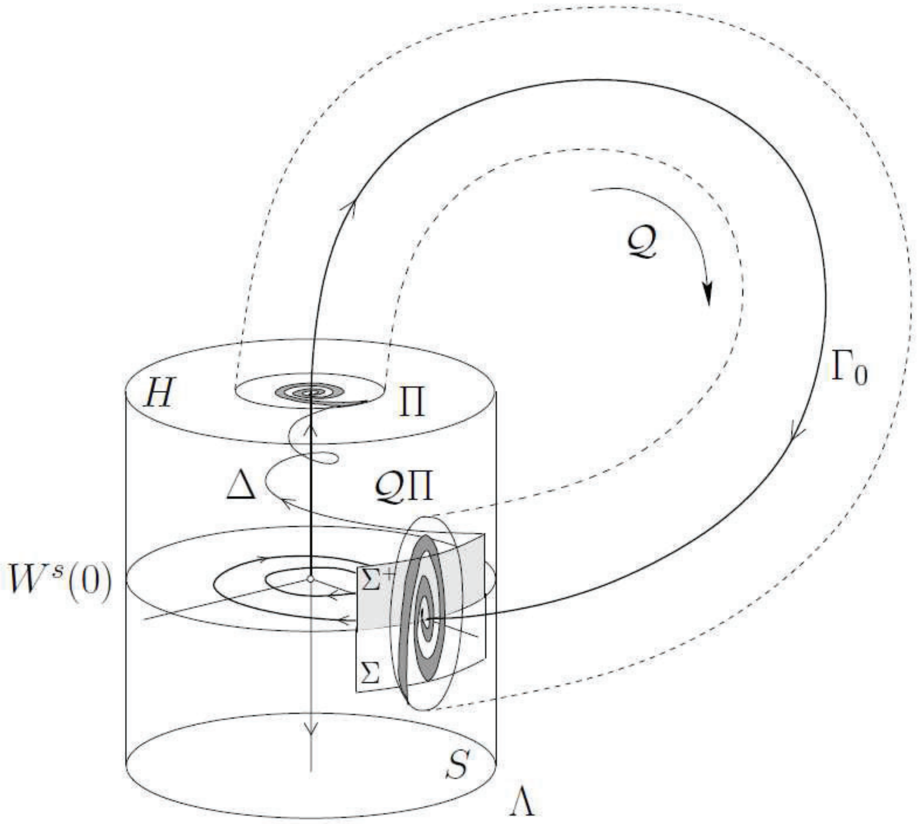
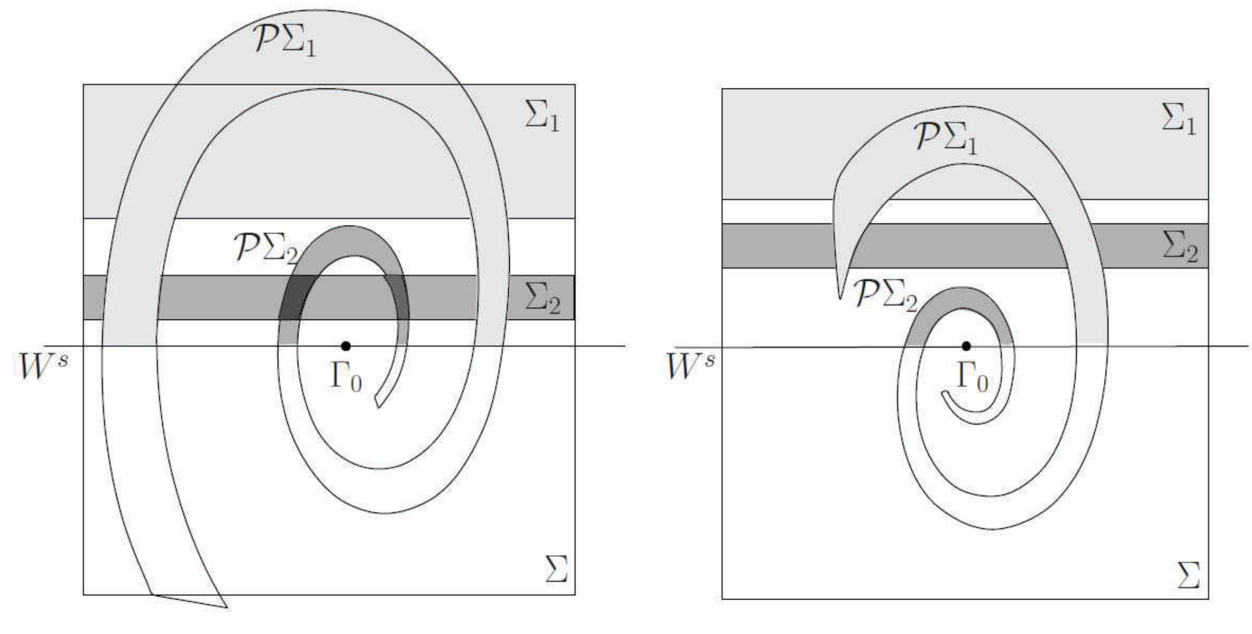
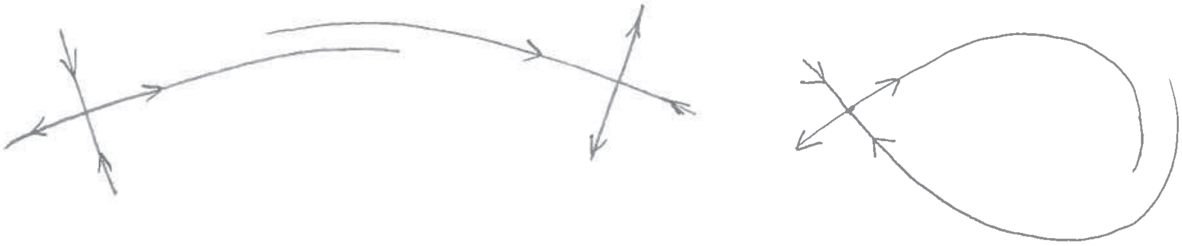


Figure 7.17: Construction of the Poincaré map $\mathcal{P}$ in the saddle-focus case.

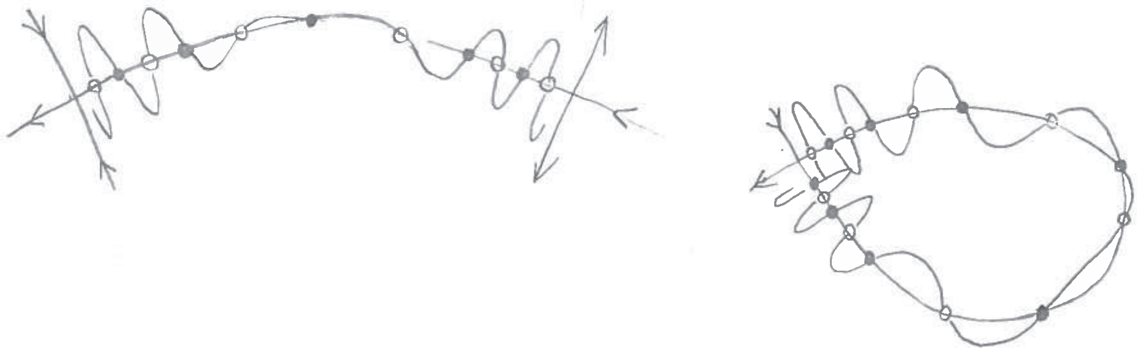


Homoclinic and heteroclinic connections and bifurcations in d.t. (reversible) systems



no intersection (structurally stable)

tangent contacts (structurally unstable)



transversal intersections (structurally stable)

Homoclinic / heteroclinic bifurcation: the tangent contact between a pair of homoclinic / heteroclinic connections

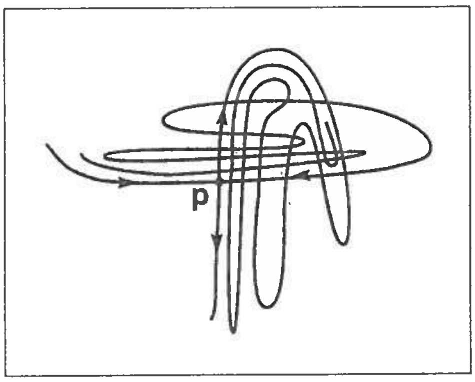


Figure 10.10 Tangle of stable and unstable manifolds implied by homoclinic points.
If the stable and unstable manifolds of a fixed-point saddle or periodic point **p** cross in one homoclinic point, then they cross infinitely many times: each forward and backward iterate of a homoclinic point is a homoclinic point.

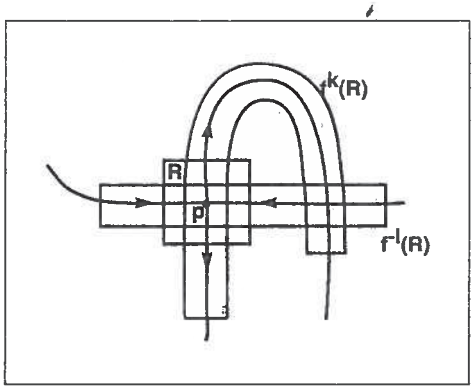


Figure 10.11 Construction of a horseshoe near a homoclinic point.
The stable and unstable manifolds of a saddle **p** intersect in a homoclinic point **x**. A rectangle R is centered at **p**. Then for some positive integers k and l, k forward iterates of R and l backward iterates of R intersect at **x**, so that $\mathbf{f}^{k+l}$ forms a horseshoe.