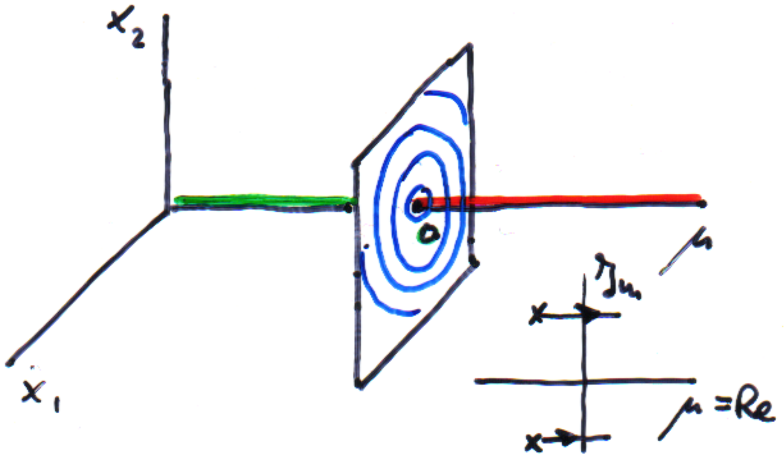
Hopf bifurcation

Linear system

$$\dot{x}_1 = \mu x_1 + \omega x_2$$

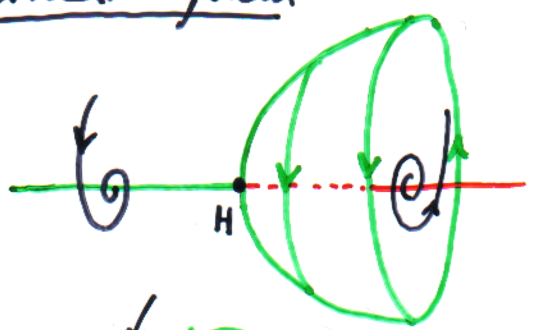
$$\dot{x}_2 = -\omega x_1 + \mu x_2$$

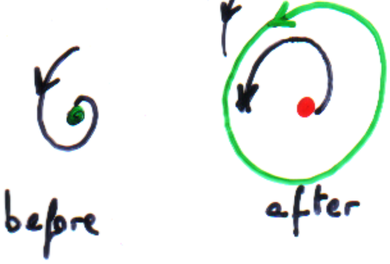
$$A = \begin{vmatrix} \mu & \omega \\ -\omega & \mu \end{vmatrix} \qquad \lambda_{1,2} = \mu \mp i\omega$$



for $\mu = 0$ we have a bifurcation because for $\mu = 0$ there are ∞ cycles why for $\mu \neq 0$ there is only one equil.

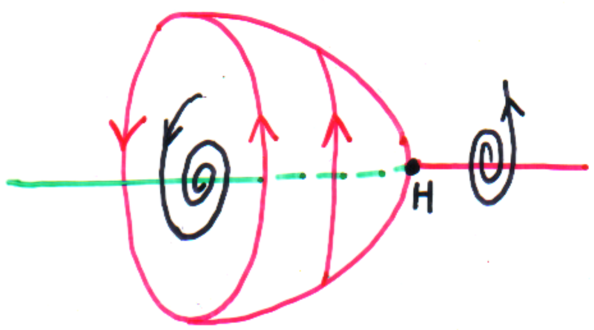
Nonlinear system

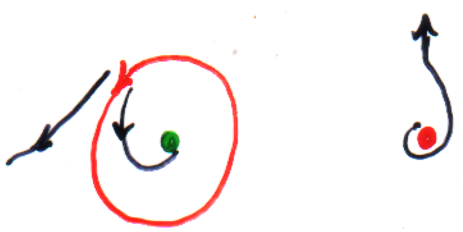




non catastrophic Hopf bifurcation

- from the right : a stable cycle shrinks and collide with an unstable equilibrium
- from the left : a stable equilibrium becomes unstable and surrounded by a limit cycle
- examples : a branch in a stream
  Parkinson's disease
  a ball in a rotating cup
  a caravan on the freeway





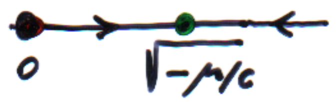
catastrophic Hopf bifurcation

- from the right : an unstable equilibrium becomes stable and surrounded by an unstable limit cycle

Normal form and Hopf theorem

$$\boxed{\begin{aligned}\dot\rho &= \mu\rho + c\rho^3\\ \dot\vartheta &= \omega\end{aligned}} \quad\Longrightarrow\quad \begin{aligned}\dot x_1 &= \mu x_1 + \omega x_2 + c x_1(x_1^2+x_2^2)\\ \dot x_2 &= -\omega x_1 + \mu x_2 + c x_2(x_1^2+x_2^2)\end{aligned}$$

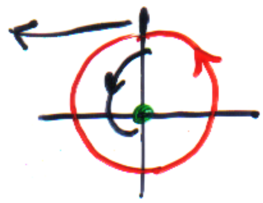
$\dot\rho = 0 \Rightarrow \begin{cases}\rho = 0 & \text{equilibrium at the origin}\\ \rho = \sqrt{-\mu/c} & \text{limit cycle (provided } \mu/c < 0)\end{cases}$



$\mu > 0$
$c < 0$

stable cycle
(non catastrophic H.)
(supercritical H.)

$\mu < 0$
$c > 0$



unstable cycle
(catastrophic H.)
(subcritical H.)

Hopf theorem

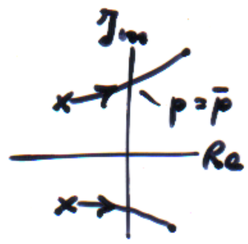
$$\dot x_1 = \mu(p)\, x_1 + \omega(p)\, x_2 + f_1(x_1\, x_2\, p)$$

$$\dot x_2 = -\omega(p)\, x_1 + \mu(p)\, x_2 + f_2(x_1\, x_2\, p)$$

$$f_i(0\,0\,p) = 0$$

$$J(p) = \begin{vmatrix}\mu(p) & \omega(p)\\ -\omega(p) & \mu(p)\end{vmatrix} \qquad \lambda_{1,2}(p) = \mu(p) \mp i\,\omega(p)$$

$$\left.\begin{aligned} tr\, J(\bar p) = 0 &\Rightarrow \mu(\bar p) = 0\\ \det J(\bar p) > 0 &\quad \text{OK}\\ \frac{d}{dp}\left[tr\, J(p)\right]_{p=\bar p} &\neq 0\end{aligned}\right\} \Rightarrow$$



$$c = \frac{1}{16}\left[\frac{\partial^3 f_1}{\partial x_1^3} + \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} + \cdots\right] \neq 0$$

$\Rightarrow$ $\exists$ unique limit cycle for p close to $\bar p$

$c > 0$ unst. cycle

$c < 0$ stable cycle

the radius of the cycle increases as $\sqrt{|p - \bar p|}$

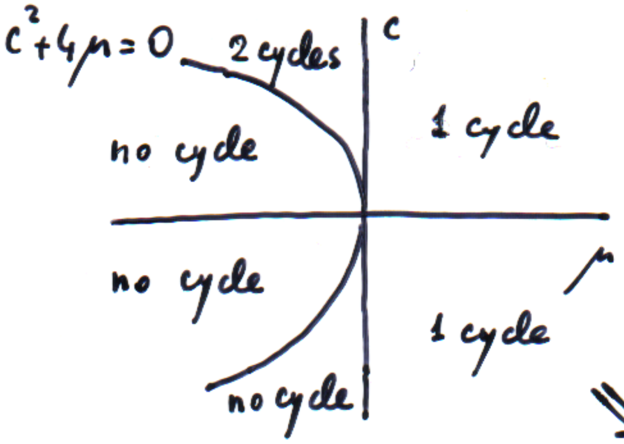
Degenerate Hopf bifurcation

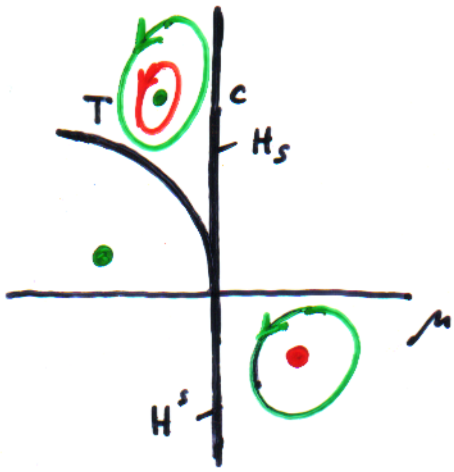
$$\dot\rho = \mu\rho + c\rho^3 - \rho^5 \quad \Rightarrow \quad \dot\rho = \rho\left(\mu + c\rho^2 - \rho^4\right)$$

$$\dot\vartheta = \omega$$

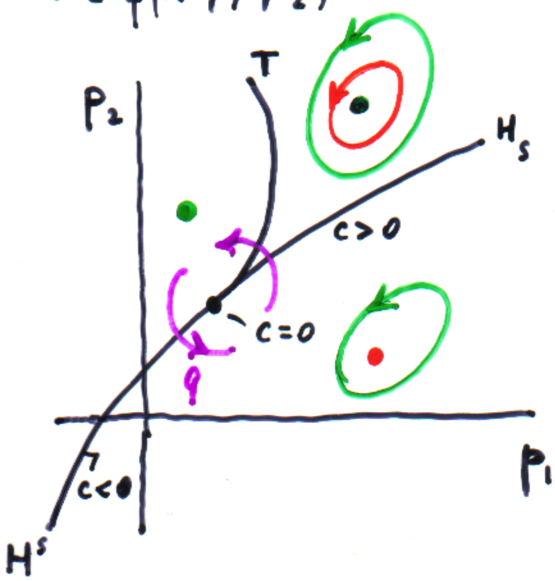
$$\Downarrow$$

$$\bar\rho = \sqrt{\frac{c \mp \sqrt{c^2+4\mu}}{2}}$$

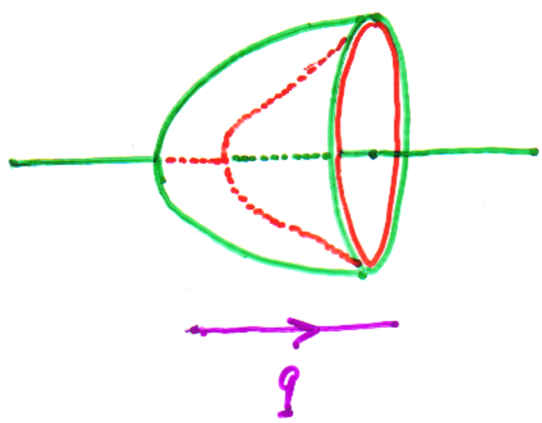




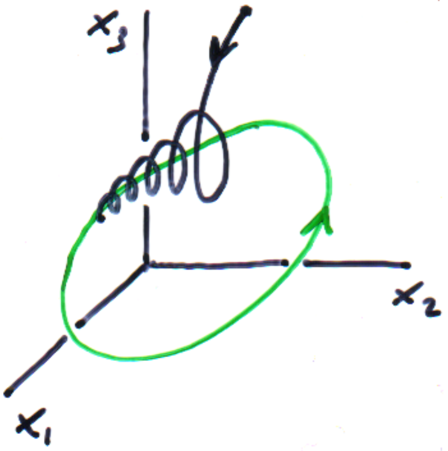
$$\dot x = f(x\ p_1\ p_2)$$

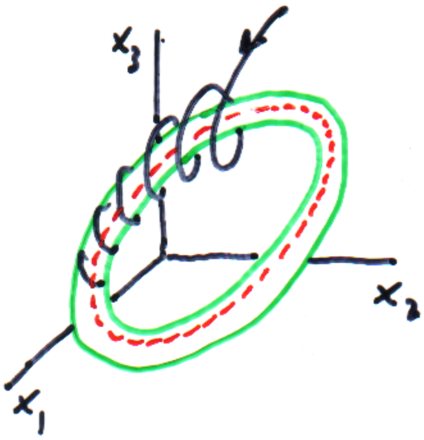


$\Rightarrow$



Neimark - Sacker bifurcation

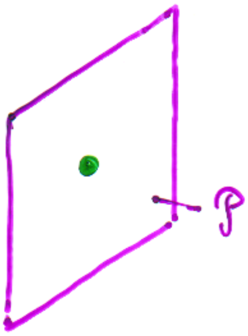


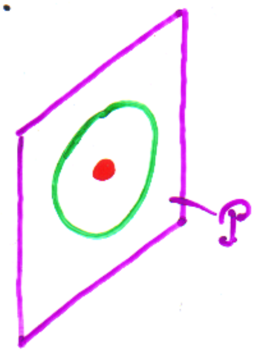


before $(p = \bar{p} - \varepsilon)$ after $(p = \bar{p} + \varepsilon)$

From the right : collision of a shrinking torus with a cycle (inside the torus)

On the Poincaré section :



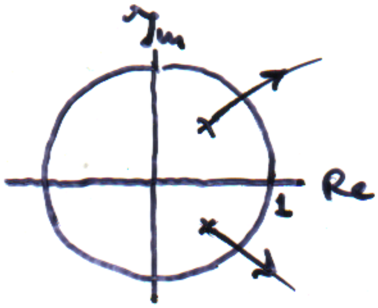


For $p = \bar{p}$ the Jacobian of the Poincaré map has complex eigenvalues with unitary module

$$\lambda_{1,2} = e^{\mp i \vartheta} \qquad \text{with} \quad \frac{\vartheta}{2\pi} = \text{irrational}$$

$\Downarrow$

torus

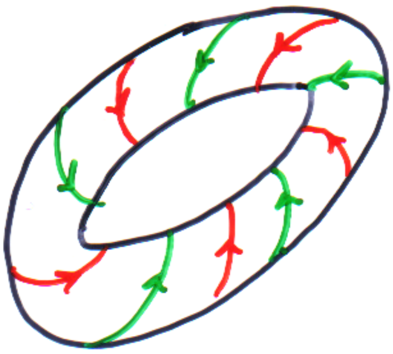


Frequency locking

Assume that $p = \bar{p} \Rightarrow$ Neimark - Sacker $\Rightarrow \lambda = e^{\mp i\vartheta}$

If $\frac{\vartheta}{2\pi} = \frac{p}{q}$ (= rational) after q returns on the Poincare' map we are back at the same point because $q\vartheta = p\,2\pi$: in other words there is a cycle on the torus (raising from the cycle through a Neimark - Sacker bifurcation).

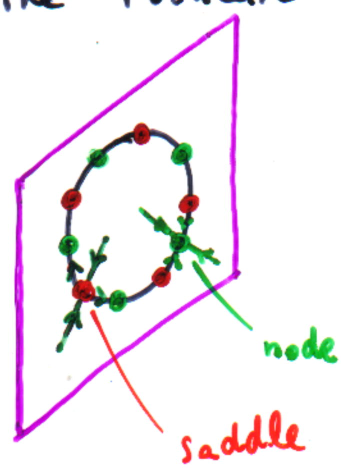
The cycle remains on the torus in the vicinity of the Neimark - Sacker bifurcation : frequency locking

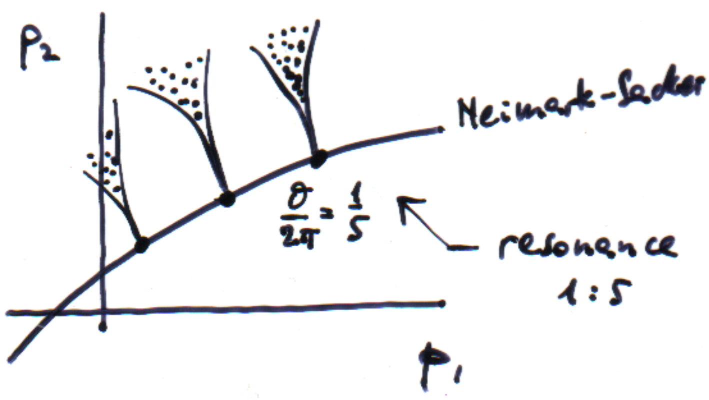


all trajectories on torus tend toward the stable cycle on the torus : this situation disappears through a tangent bifurcation of cycles

$\Downarrow$

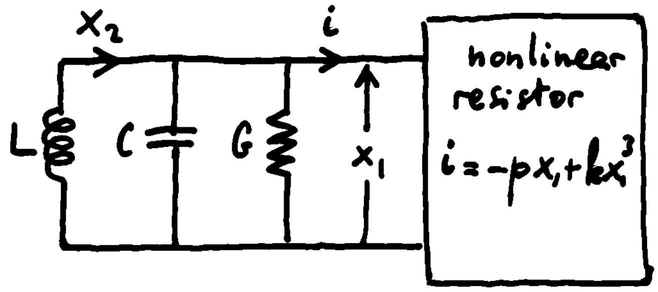
On the Poincare' section

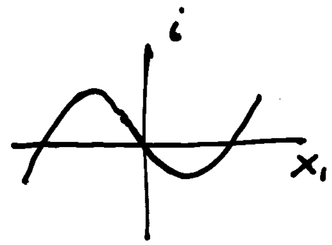




Problems

P. 1 (3) – Consider the following electrical network





and determine the value $\bar{p}$ corresponding to a Hopf bifurcation. Then, show the equilibria and the cycle of the system in the space (x_1, x_2, p).

P. 2 (3) – Consider the prey-predator model

$$\begin{cases} \dot{x}_1 = r x_1 \left(1 - \dfrac{x_1}{K}\right) - a \dfrac{x_1}{b + x_1} x_2 \\ \dot{x}_2 = e a \dfrac{x_1}{b + x_1} x_2 - m x_2 \end{cases}$$

and determine the relationship among the parameters giving rise to a Hopf bifurcation.

(Hint: write the state equations in the form

$$\dot{x}_1 = x_1 F_1(x_1, x_2)$$

$$\dot{x}_2 = x_2 F_2(x_1)$$

and then compute the Jacobian matrix and use the inequalities

$$\frac{\partial F_1}{\partial x_2} < 0 \qquad \frac{\partial F_2}{\partial x_1} > 0$$