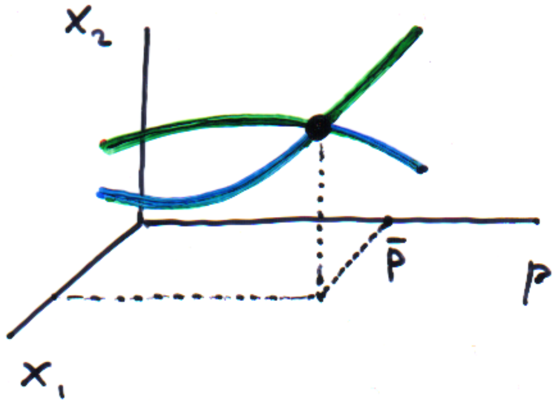
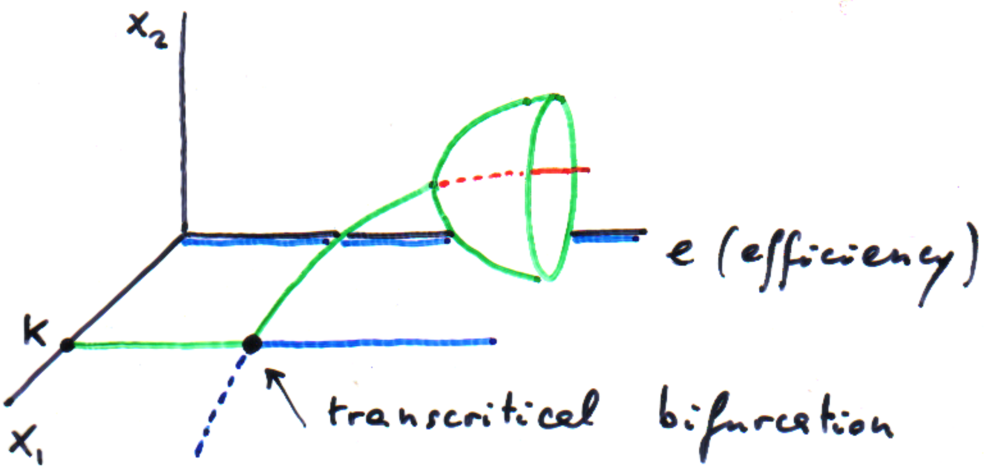
# Transcritical bifurcation (1)

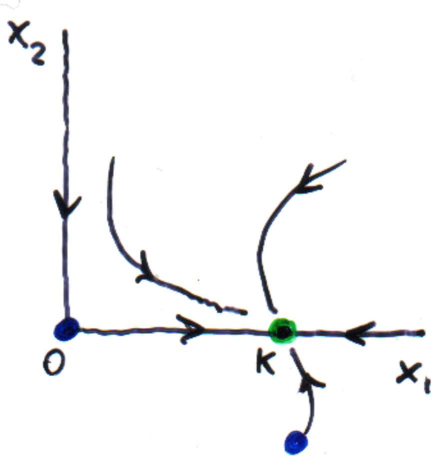



For any $p$ there are two equilibria: one stable and one unstable. The two equilibria collide at $\bar{p}$ where they exchange stability

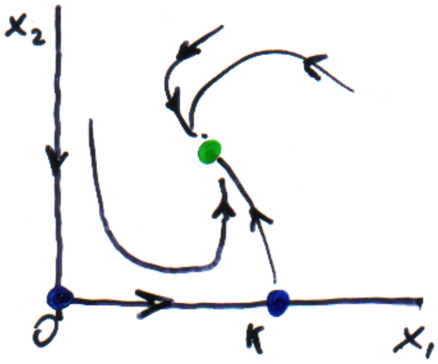
## Ex. 1 Prey-predator model (see Ex. 8 Lecture 1)

$$\begin{cases} \dot{x}_1 = r x_1 \left(1 - \dfrac{x_1}{K}\right) - a \dfrac{x_1}{b + x_1} x_2 & \text{prey} \\ \dot{x}_2 = e a \dfrac{x_1}{b + x_1} x_2 - m x_2 & \text{predator} \end{cases}$$




$e < e_{tc}$




$e > e_{tc}$

# Transcritical bifurcation (2)

Transcritical bifurcations (of equilibria) can occur in systems of any order : even in first order systems

The simplest form of a bifurcation is called <u>normal form</u>

The normal form of the transcritical bifurcation is the following first order system

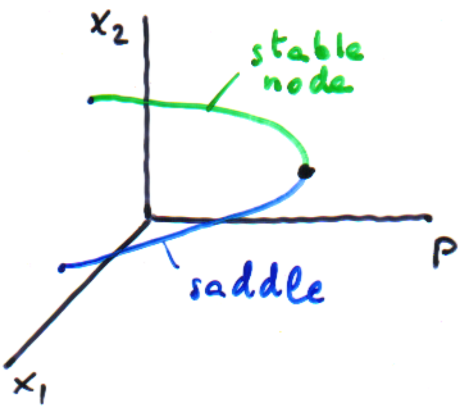
$$\boxed{\dot{x} = p x - x^2}$$

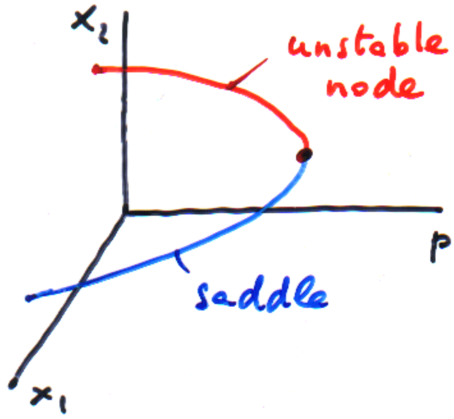
Equilibria : $\dot{x} = 0 \Rightarrow \begin{cases} \bar{x} = 0 \\ \bar{x} = p \end{cases}$

Stability : $J = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \left. p - 2x \right|_{\bar{x}} = \begin{cases} p & \text{stable} \Leftrightarrow p<0 \\ -p & \text{stable} \Leftrightarrow p>0 \end{cases}$


- For $p=0$ we have only one equilibrium , while for $p \neq 0$ we have two equilibria $\Rightarrow$ for $p=0$ the system is not structurally stable $\Rightarrow$ $p=0$ is a bifurcation
- For $p=0$ we have a collision of equilibria $\Rightarrow$ bifurcation
- For $p=0$ there are eigenvalues on the stability boundary

# Saddle - node bifurcation




The node can be either stable or unstable : in the first case we have a catastrophic transition.

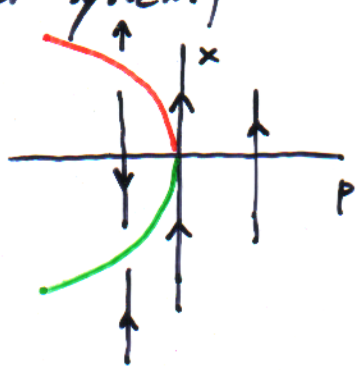
When $p = \bar{p}$ the Jacobian matrices of the two equilibria must have the same eigenvalues, while for $p < \bar{p}$ the node has eigenvalues of the same sign, and the saddle has positive and negative eigenvalues. This implies that for $p = \bar{p}$ one eigenvalue of the node and one eigenvalue of the saddle hits the stability boundary.

<u>Examples</u> avalanches
earth-quakes

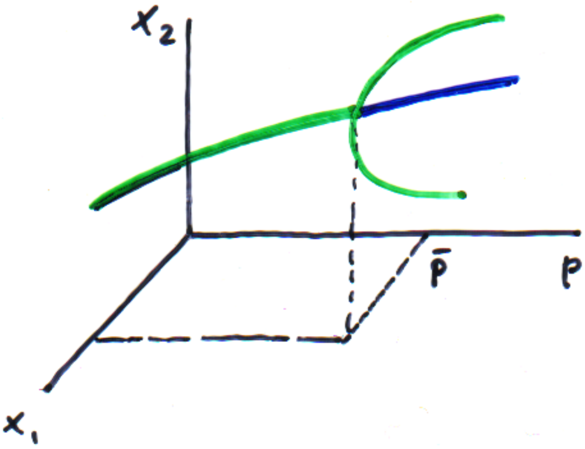
<u>Terminology</u> the saddle - node bifurcation is also called <u>fold</u> or <u>tangent</u> bifurcation

<u>Normal form</u> (first order system)

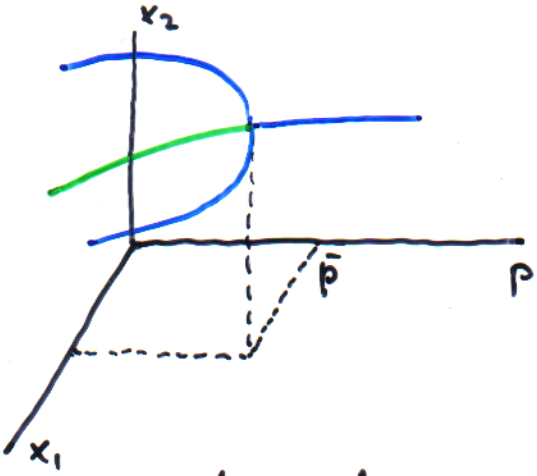
$$\boxed{\dot{x} = p + x^2}$$



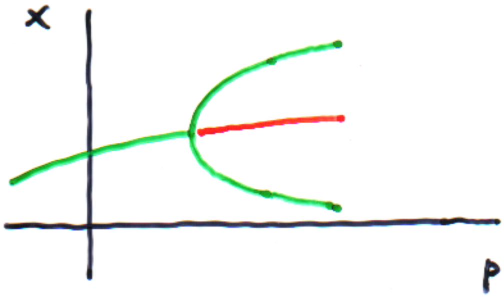
# Pitchfork bifurcation

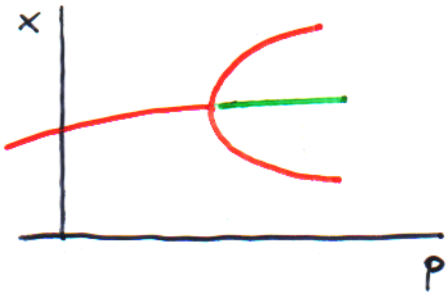



non catastrophic

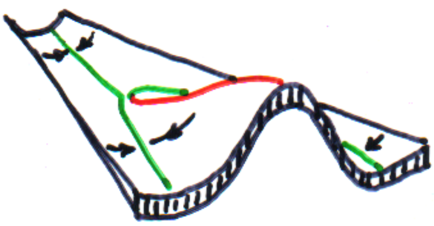



catastrophic



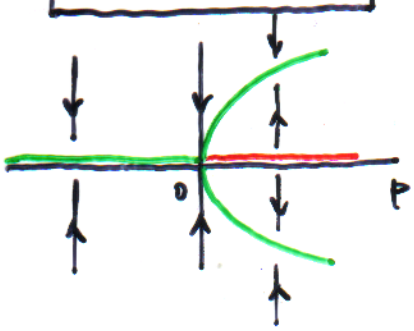




## Example



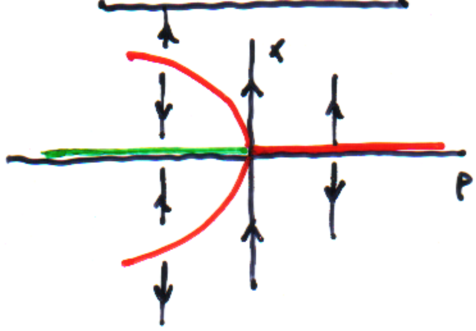
## Normal forms

$$\boxed{\dot{x} = px - x^3}$$




non catastrophic
supercritical

$$\boxed{\dot{x} = px + x^3}$$




catastrophic
subcritical

# From equilibria to cycles

## Transcritical in $\mathbb{R}^3$

the two cycles exchange their stability

## Saddle-node in $\mathbb{R}^2$

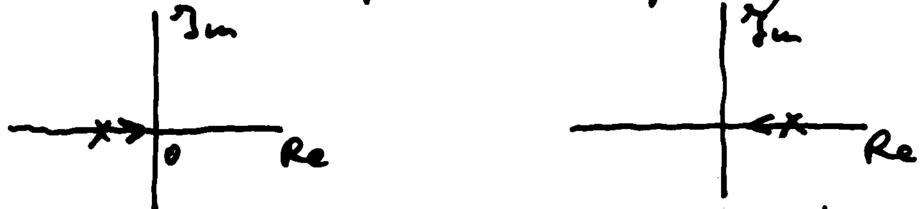
the two cycles disappear

## Pitchfork in $\mathbb{R}^2$

before after

# Problems

**P. 1 [(1)] –** Verify (using the normal form) that the saddle-node bifurcation is characterized by two zero eigenvalues (one associated with one equilibrium and one with the other equilibrium) coming from opposite sides of the imaginary axis, i.e.



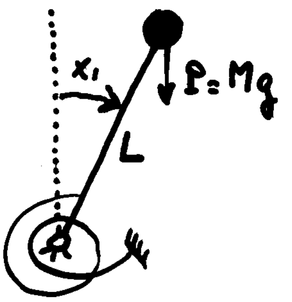

1-st equil. 2-nd equil.

**P. 2 [(2)] –** Consider the following 1-st order system

$$\dot{x} = r x \left(1 - \frac{x}{K}\right) - h$$

where $x$ is the amount of resource and $h$ is the harvest rate. Assume that $h$ is constant in time and discuss the behavior of the system for all $h \geq 0$. Show that the system has two types of behavior depending upon the value of $h$. Find the critical value of $h$ and determine the kind of bifurcation involved.

**P. 3 [(2)] –** Show that the mechanical system described in the figure is described by




$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{J}\left(PL \operatorname{sen} x_1 - K x_1 - H x_2\right) \end{cases}$$

where $K x_1$ is the momentum of the spring and $H$ is a friction coefficient. Study the equilibria of the system for small values of $P$ and then consider higher and higher values of $P$. Prove that a pitchfork bifurcation occurs.