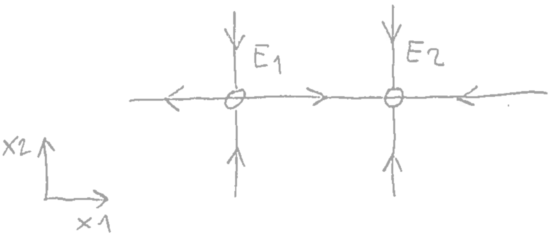
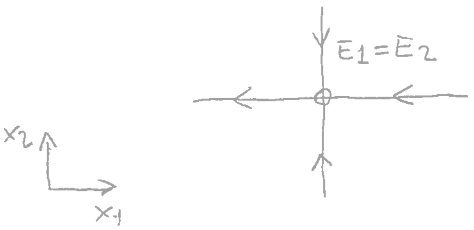
Algebraic characterization of the transcritical, saddle-node, and pitchfork bifurcations





→ two equilibria, E1 and E2, collide along with change in one parameter p

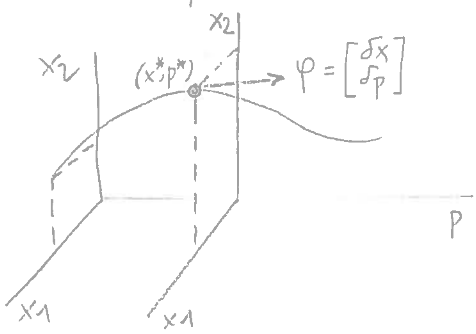
→ before the collision, one eigenvalue, λ_i, is positive in E1 and negative in E2, otherwise the collision cannot occur

→ the collision occurs in the direction of the eigenvectors associated to λ_i, that align while approaching the bifurcation

→ at the bifurcation the eigenvalues of E1 and those of E2 coincide, so that

$$\lambda_i = 0 \qquad \text{(condition 1)}$$

The continuation problem $f(x,p)=0$ (for continuous time systems) defines one-dim. equilibrium branches in the $(n+1)$-dim. continuation space (x,p)

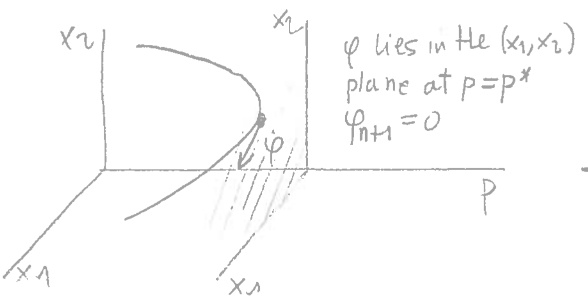


→ expanding f at a point (x^*,p^*) of an equilibrium branch

$$f(x,p) = f(x^*+\delta x, p^*+\delta p) =$$

$$= \underbrace{f(x^*,p^*)}_{0} + \left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial p}\right]_{(x^*,p^*)} \begin{bmatrix}\delta x\\ \delta p\end{bmatrix} + \cdots$$

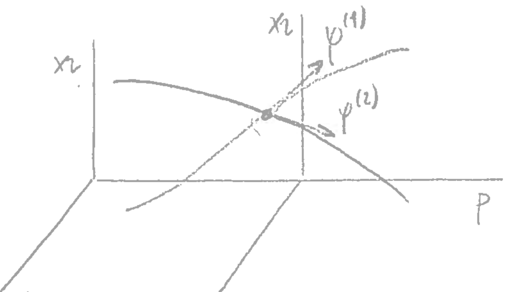
we see that the tangent direction to the eq. branch at (x^*,p^*) belongs to the null-space of the $n\times(n+1)$ Jacobian $\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial p}\right]$ evaluated at (x^*,p^*)



→ if $\mathrm{rank}\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial p}\right]_{(x^*,p^*)} = n$ (full rank)

then the tangent direction is unique.

This is the case of the saddle node bifurcation

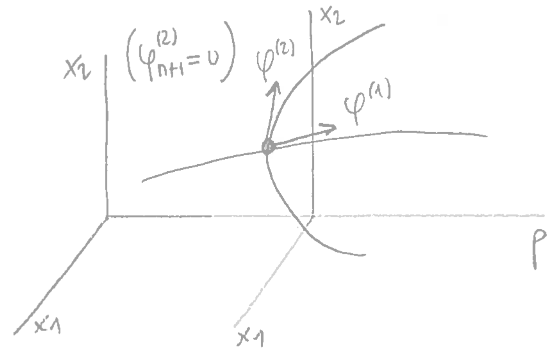


→ if $\mathrm{rank}\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial p}\right]_{(x^*,p^*)} = n-1$ (rank-defect 1, cond. 2)

then the null-space N^* of $\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial p}\right]_{(x^*,p^*)}$ is 2-dim.

→ generically, there are 2 eq. branches passing through (x^*,p^*) with tangent directions $\varphi^{(1)}, \varphi^{(2)} \in N^*$

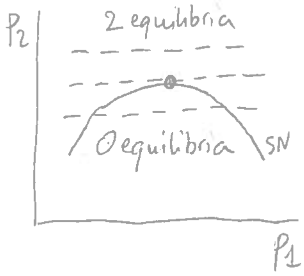
→ if both $\varphi^{(1)}_{n+1}$ and $\varphi^{(2)}_{n+1}$ (the p-components) are nonzero then the bifurcation is transcritical

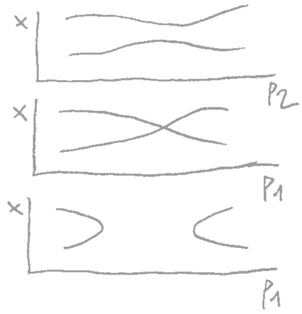


→ if $\varphi^{(1)}_{n+1}=0$ or $\varphi^{(2)}_{n+1}=0$ (condition 3)
then the bifurcation is <u>pitchfork</u>
($\varphi^{(1)}_{n+1}=\varphi^{(2)}_{n+1}=0$ would be a higher codimension)

Notes on codimension

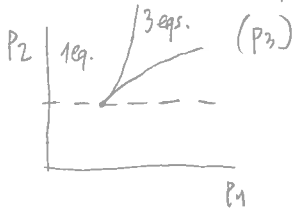
→ The transcritical bif. has codim-2.
(two critical conditions 1 + 2)
In fact, it is generically not encountered while moving one par (p_1 in the figure), except if another par (p_2) is suitably chosen.

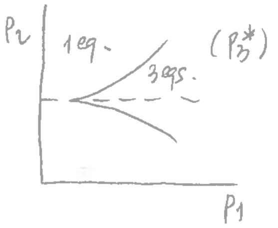


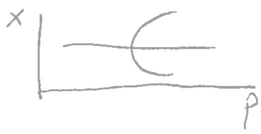


→ However, the transcritical has codim-1 when it involves an equilibrium that exists for all values of the parameters, so that it cannot disappear through a SN (e.g. the extinction equilibrium of the logistic model or the equilibrium (K, 0) of the prey-predator model). In these situations, condition 2 (the rank-defect of the Jacobian $[\frac{\partial f}{\partial y}\ \frac{\partial f}{\partial p}]$ at the bif) is "built-in" in the system's equations.

→ The pitchfork bif. has codim-3.
(three critical conditions 1+2+3)
In fact, it is generically not encountered while moving one par (p_1) even if a second par (p_2) is suitably chosen. A third par (p_3) should indeed be tuned so that it is possible to enter the cusp while moving p_1







→ However, the pitchfork has codim-1 when it involves an equilibrium that exists for all paramter values in a system with a symmetry, so that the equilibrium cannot disappear and the other branch involves two symmetric equilibria that are present on the same side of the bifurcation (e.g. a trivial rest point in a mechanical system that loses stability with the appearance of two new symmetric stable rest points). In these situations conditions 2 and 3 are "built-in" in the system's equations (cond. 2 due to the existence of the trivial equilibrium and cond. 3 due to the symmetry).

Center manifold reduction and normal forms

System (c.t.): $\dot{x} = f(x, p)$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}$

Normal form: $\dot{z} = f_n(z, \alpha)$, $z \in \mathbb{R}^{n_c}$, $\alpha \in \mathbb{R}$

Consider a local bifurcation of the equilibrium $\bar{x}$ at $p = \bar{p}$, with

n_s : # stable eigenvalues ($\mathrm{Re}\,\lambda_i < 0$), $\mathbb{X}_s$: subspace of associated (standard and generalized) eigenvectors (dim $= n_s$)

n_c : # critical eigenvalues ($\mathrm{Re}\,\lambda_i = 0$), $\mathbb{X}_c$: ... (dim $= n_c$)

n_u : # unstable eigenvalues ($\mathrm{Re}\,\lambda_i > 0$), $\mathbb{X}_u$: ... (dim $= n_u$)

Stable, center, and unstable manifolds

At $p = \bar{p}$ there are three (locally defined) invariant manifolds passing through $\bar{x}$

stable manifold $\mathcal{X}_s$: dim $= n_s$, tangent to $\mathbb{X}_s$ at $\bar{x}$
dynamics on $\mathcal{X}_s$ equivalent to the linearized dyn. on $\mathbb{X}_s$

center manifold $\mathcal{X}_c$: dim $= n_c$, tangent to $\mathbb{X}_c$ at $\bar{x}$
dynamics on $\mathcal{X}_c$ critically dependent on nonlinearities

unstable manifold $\mathcal{X}_u$: dim $= n_u$, tangent to $\mathbb{X}_u$ at $\bar{x}$
dynamics on $\mathcal{X}_u$ equivalent to the linearized dyn. on $\mathbb{X}_u$

Parametrized center (or "slow") manifold

For p close to $\bar{p}$ it is possible to define (locally to $\bar{x}(\bar{p})$, the equilibrium $\bar{x}(p)$ might not exist on one side of the bifurcation) a family $\mathcal{X}_c(p)$ of invariant manifolds on which the bifurcation takes place.

$\mathcal{X}_c(p)$: dim $= n_c$, tangent to $\mathbb{X}_c(p)$ at $\bar{x}(p)$ (whenever the equilibrium exists)

$\mathbb{X}_c(p)$: subspace generated by the (standard and generalized) eigenvectors associated to the λ_i that are critical at the bifurcation.

dynamics on $\mathcal{X}_c(p)$ equivalent to the dynamics of the normal form

[under the change of variable and parameter $x = H(z, \alpha)$, $p = V(\alpha)$, with $\bar{x}(\bar{p}) = H(0,0)$, $\bar{p} = V(0)$, H continuous and invertible in z close to $(z, \alpha) = (0,0)$, V continuous and invertible close to $\alpha = 0$]

Notes

- the dynamics on $\mathcal{X}_c(p)$ for p close to $\bar{p}$ is "slow" ($|\mathrm{Re}\,\lambda_i|$ small), compared to the dynamics transversal to $\mathcal{X}_c(p)$ ($|\mathrm{Re}\,\lambda_i|$ finitely large)
- the dynamics transversal to $\mathcal{X}_c(p)$ is ruled by the linearized dyn. in the subspace of the non-critical eigenvectors.

Parameterized Homological eq.

$x(t) = H(z(t), \alpha)$, invariance of the (parameterized) center manifold for $t \geqslant 0$

$$\Big\downarrow \, d/dt$$

$$f(\underbrace{H(z,\alpha)}_{x}, \underbrace{V(\alpha)}_{p}) = H_z(z,\alpha)\, \underbrace{G(z,\alpha)}_{\dot{z}}$$

to be expanded with respect to (z, α) around $(0,0)$.

Note: solving for the coefficients of the expansions of H, V, G leaves some freedom, due to the non-uniqueness of the (p-) center manifold. The freedom is exploited to "simplify" G as much as possible.

Note: the expansion is truncated to the lowest orders of z and α such that h.o.t. are proved to be dominated. This means proving that the systems' family $\dot{z} = G(z,\alpha)$ is equivalent to the truncated family $\dot{z} = f_n(z,\alpha)$, i.e. to the normal form [for any α close to $\alpha = 0$, system $G(z,\alpha)$ must be equivalent to system $f_n(z,\alpha)$].