Quasi periodicity

$$y(t) = A \operatorname{sen} \omega_1 t + B \operatorname{sen} \omega_2 t$$

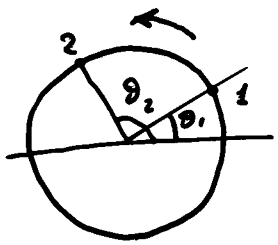
$$\frac{\omega_1}{\omega_2} = \begin{cases} \text{rational} = \frac{p}{q} & (\text{probability} = 0) \\ \text{irrational} & (\text{probability} = 1) \end{cases}$$

$$\frac{\omega_1}{\omega_2} = \frac{p}{q} = \frac{T_2}{T_1} \Rightarrow p\,T_1 = q\,T_2 \Rightarrow y(\cdot) \text{ is periodic of period } T$$

$$\omega_i = \frac{2\pi}{T_i} \qquad T = p\,T_1 = q\,T_2$$

$$\frac{\omega_1}{\omega_2} = \text{irrational} \Rightarrow y(\cdot) \text{ is } \underline{\text{quasiperiodic}}$$

Ex. 1. Two runners

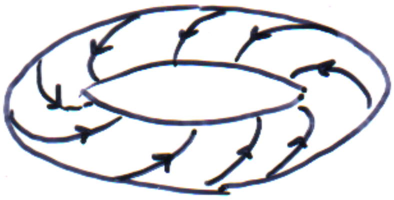


$$\dot\vartheta_1 = \omega_1$$
$$\dot\vartheta_2 = \omega_2$$

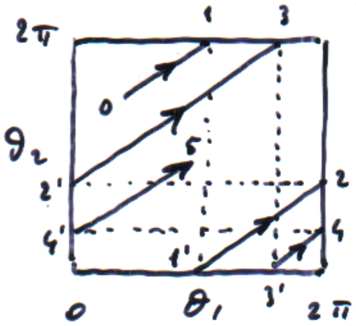
If $\frac{\omega_1}{\omega_2} = \frac{p}{q}$ runner 1 will complete p tours when runner 2 will complete q tours

Thus, after a period $T = p\,\frac{2\pi}{\omega_1} \left(= q\,\frac{2\pi}{\omega_2} \right)$ the two runners will be in the same conditions.

Visualization on torus



2-torus



$$\frac{d\vartheta_2}{d\vartheta_1} = \frac{\dot{\vartheta}_2}{\dot{\vartheta}_1} = \frac{\omega_2}{\omega_1}$$

If $\dfrac{\omega_1}{\omega_2} = \dfrac{p}{q}$ we have a cycle on torus.

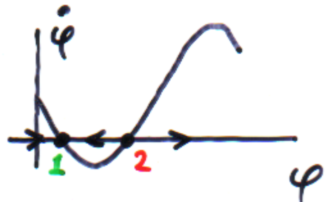
If $\dfrac{\omega_1}{\omega_2}$ is irrational we have a trajectory on torus that never comes back to the initial conditions but densely covers the entire torus.

Ex. 2 Two friends running (phase locking)

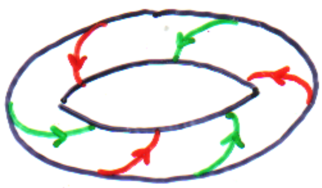
$$\begin{cases} \dot{\vartheta}_1 = \omega_1 + K_1 \sin(\vartheta_2 - \vartheta_1) \\ \dot{\vartheta}_2 = \omega_2 + K_2 \sin(\vartheta_1 - \vartheta_2) \end{cases} \qquad \varphi = \vartheta_1 - \vartheta_2$$

$$\dot{\varphi} = \dot{\vartheta}_1 - \dot{\vartheta}_2 = \omega_1 - \omega_2 - (K_1 + K_2)\sin\varphi$$

$$\dot{\varphi} = 0 \iff \sin\bar{\varphi} = \frac{\omega_1 - \omega_2}{K_1 + K_2}$$



$|\omega_1 - \omega_2| < K_1 + K_2$

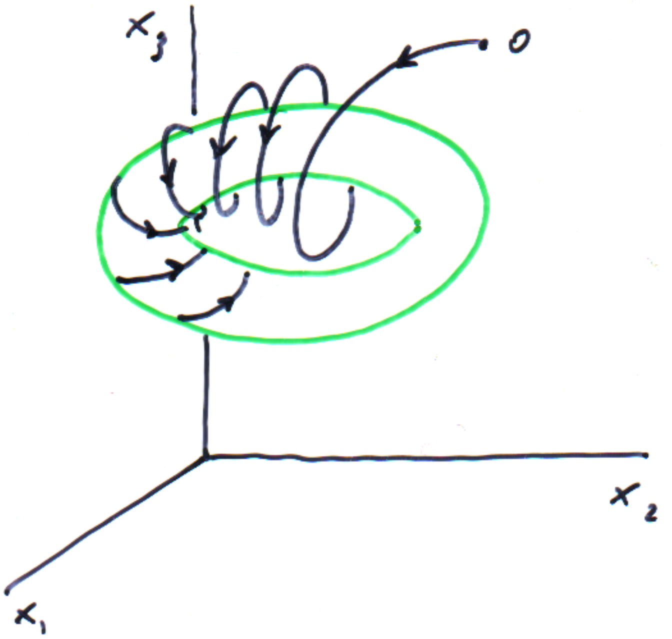


$$\omega^* = \lambda\omega_1 + (1-\lambda)\omega_2$$

$$\lambda = \frac{K_2}{K_1 + K_2}$$

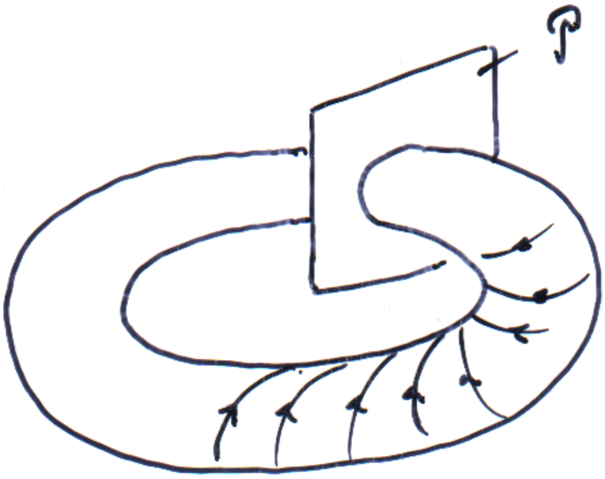
(If $|\omega_1 - \omega_2| > K_1 + K_2$ the behaviour is quasiperiodic)

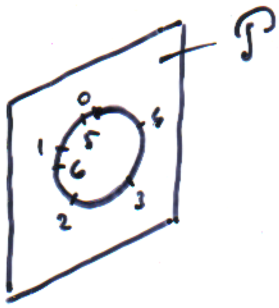
Third order systems



The trajectory starting from 0 tends toward the torus.

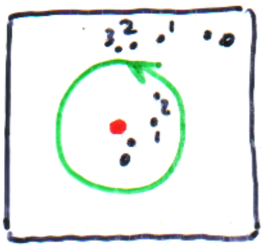
If the same happens for all points 0 close to the torus (also inside it) we say that the torus is an attractor



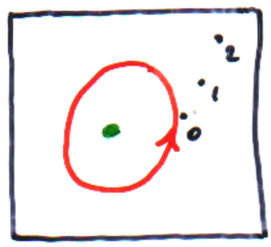


Poincaré section

Inside the torus there is a limit cycle



stable torus



unstable torus

In higher order systems we can also have <u>saddle tori</u>.

Non existence of tori

Theorem 1 (non existence condition)

If the divergence $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \dots + \frac{\partial f_n}{\partial x_n}$ does not change sign in a domain $\Omega \subset R^n$ (and, at most, annihilates on a manifold of dimension $n-1$) then there are no tori in Ω.

Proof. Same arguments used in Theorem 4 of lecture 6.

Ex. 3 (Lorenz system)

$$\dot{x}_1 = \sigma (x_2 - x_1)$$

$$\dot{x}_2 = r x_1 - x_2 - x_1 x_3$$

$$\dot{x}_3 = x_1 x_2 - b x_3$$

$$\text{divergence} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -\sigma - 1 - b < 0$$

Thus, in the Lorenz system trajectories converge (because volumes contract) so that there cannot be an attracting or repelling torus (because inside the torus there should be a repelling or attracting cycle).

Problems

(2)
P. 1. Reconsider the system (see Ex. 2)

$$\dot{\vartheta}_1 = \omega_1 + k_1 \sin(\vartheta_2 - \vartheta_1)$$
$$\dot{\vartheta}_2 = \omega_2 + k_2 \sin(\vartheta_1 - \vartheta_2)$$

Find a "conserved quantity" for this system (i.e. a sort of energy function V such that $V = \text{const.}$, i.e. $\dot{V} = 0$).

(3)
P. 2. Consider the equations

$$m\ddot{\rho} = \frac{h^2}{m\rho^3} - k \qquad h = \text{const.} > 0$$
$$\dot{\vartheta} = \frac{h}{m\rho^2}$$

These equations describe (in polar coordinates) the motion of a mass m subject to a central force of constant strength $k > 0$.

(i) Show that the system has a solution
$$\rho = R \qquad \dot{\vartheta} = \omega$$
corresponding to uniform circular motion.

(ii) Find the frequency ω_r of small radial oscillations about the circular orbit

(iii) Show by a geometric argument that the motion is either periodic or quasiperiodic.