Ex1 : Two runners

$$\dot\rho_1 = \rho_1(1-\rho_1^2) \;,\quad \dot\rho_2 = \rho_2(1-\rho_2^2) \;,\quad x_1 = \rho_1\cos\vartheta_1 \,,\; x_2 = \rho_1 \sin\vartheta_1$$
$$\dot\vartheta_1 = \omega_1 \;,\quad \dot\vartheta_2 = \omega_2 \;,\quad x_3 = \rho_2\cos\vartheta_2 \,,\; x_4 = \rho_2\sin\vartheta_2$$

$$\begin{aligned}
\dot x_1 &= x_1 - \omega_1 x_2 - (x_1^2+x_2^2)\,x_1 \\
\dot x_2 &= \omega_1 x_1 + x_2 - (x_1^2+x_2^2)\,x_2 \\
\dot x_3 &= x_3 - \omega_2 x_4 - (x_3^2+x_4^2)\,x_3 \\
\dot x_4 &= \omega_2 x_3 + x_4 - (x_3^2+x_4^2)\,x_4 \\
y &= A x_2 + B x_4
\end{aligned}
\;,\quad x(0) = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

Notes:

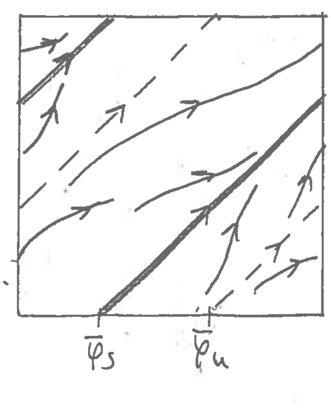
- The <u>invariant torus</u> is the 2-dimensional manifold $\{x_1^2+x_2^2=1,\; x_3^2+x_4^2=1\}$
- ω_1/ω_2 rational $\Rightarrow$ there are infinitely many stable (but not asymptotically stable) cycles on the torus. Fixing e.g. $x_1(0)=1$, $x_2(0)=0$, two different initial conditions with $x_3^2(0)+x_4^2(0)=1$ belong to two different cycles.
- ω_1/ω_2 irrational $\Rightarrow$ the dynamics on the torus is quasiperiodic. Each trajectory on the torus is aperiodic and <u>densely</u> fill the torus. Trajectories starting from close initial conditions remain close for all $t>0$.
- The torus is k-dimensional, $k \geq 2$, if there are k incommensurable frequencies (f_i/f_j irrational for any $i \neq j$, $i,j = 1,\dots,k$) in the system.

Ex 2: Two friends running

$$\dot x_1 = \dot\rho_1\cos\vartheta_1 - \rho_1\sin\vartheta_1\,\dot\vartheta_1 = \rho_1(1-\rho_1^2)\cos\vartheta_1 - \rho_1\sin\vartheta_1\big(\omega_1 + K_1\overbrace{\sin(\vartheta_2-\vartheta_1)}^{\sin\vartheta_2\cos\vartheta_1 - \sin\vartheta_1\cos\vartheta_2}\big)$$
$$= x_1 - \omega_1 x_2 - (x_1^2+x_2^2)\,x_1 - \frac{K_1 x_2}{\sqrt{x_1^2+x_2^2}\,\sqrt{x_3^2+x_4^2}}\,(x_4 x_1 - x_2 x_3)$$

Notes:

- Similarly for $\dot x_2, \dot x_3, \dot x_4$
- $|\omega_1-\omega_2| < K_1+K_2 \Rightarrow$ there are two cycles on the torus (one asymp. stable, the other unstable). The phase difference on the stable cycle is constant (phase locking)



Quasiperiodicity in discrete time

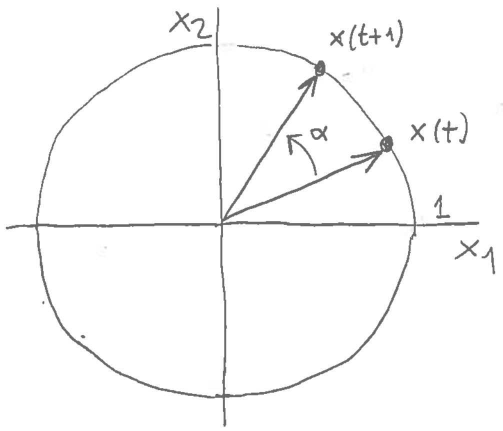
$$y(t) = A \sin(\alpha t)$$

$\alpha/\pi = $ rational (nongeneric) / irrational (generic)

$\alpha/\pi = \frac{p}{q} \Rightarrow 2q\alpha = p\,2\pi \Rightarrow y$ is periodic with period $T = 2q$

α/π irrational $\Rightarrow y$ is quasiperiodic

EX : Rotation



$x(t+1) = R\,x(t)$ (linear system)

+ h.o.t. to make the unit circle an attractive invariant curve*

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{(rotation matrix)}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y(t) = A\,x_2(t)$$

Notes :

- The invariant "torus" is the unit circle $x_1^2 + x_2^2 = 1$. It is called "torus" by analogy with the continuous-time case. It is an invariant curve.
- α/π rational $\Rightarrow$ there are infinitely many stable (not asymp. stable) cycles. E.g. two different initial conditions on the unit circle belong to to two different cycles.
- α/π irrational $\Rightarrow$ the dynamics on the torus is quasiperiodic (same comments as in continuous time).
- The torus is K-dimensional, $K \geq 1$, if there are K rotation angles incommensurable with π and among them (α_i/π, α_i/α_j irrationals for all $i \neq j$, $i,j = 1,\dots,K$)

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* for example

$$\rho(t+1) = \rho(t) + \frac{1-\rho(t)}{2} = \frac{1+\rho(t)}{2} \Rightarrow \bar{\rho} = 1 \text{ is globally stable}$$

$$\theta(t+1) = \theta(t) + \alpha$$

$$x_1(t+1) = \rho(t+1)\cos\theta(t+1) = \frac{1}{2}(1+\rho(t))(\cos\theta(t)\cos\alpha - \sin\theta(t)\sin\alpha)$$

$$= \frac{1}{2}(x_1(t)\cos\alpha - x_2(t)\sin\alpha) + \frac{1}{2\sqrt{x_1^2(t)+x_2^2(t)}}(x_1(t)\cos\alpha - x_2(t)\sin\alpha)$$

$$x_2(t+1) = \dots = \frac{1}{2}(x_1(t)\sin\alpha + x_2(t)\cos\alpha)\left(1 + \frac{1}{\sqrt{x_1^2(t)+x_2^2(t)}}\right)$$

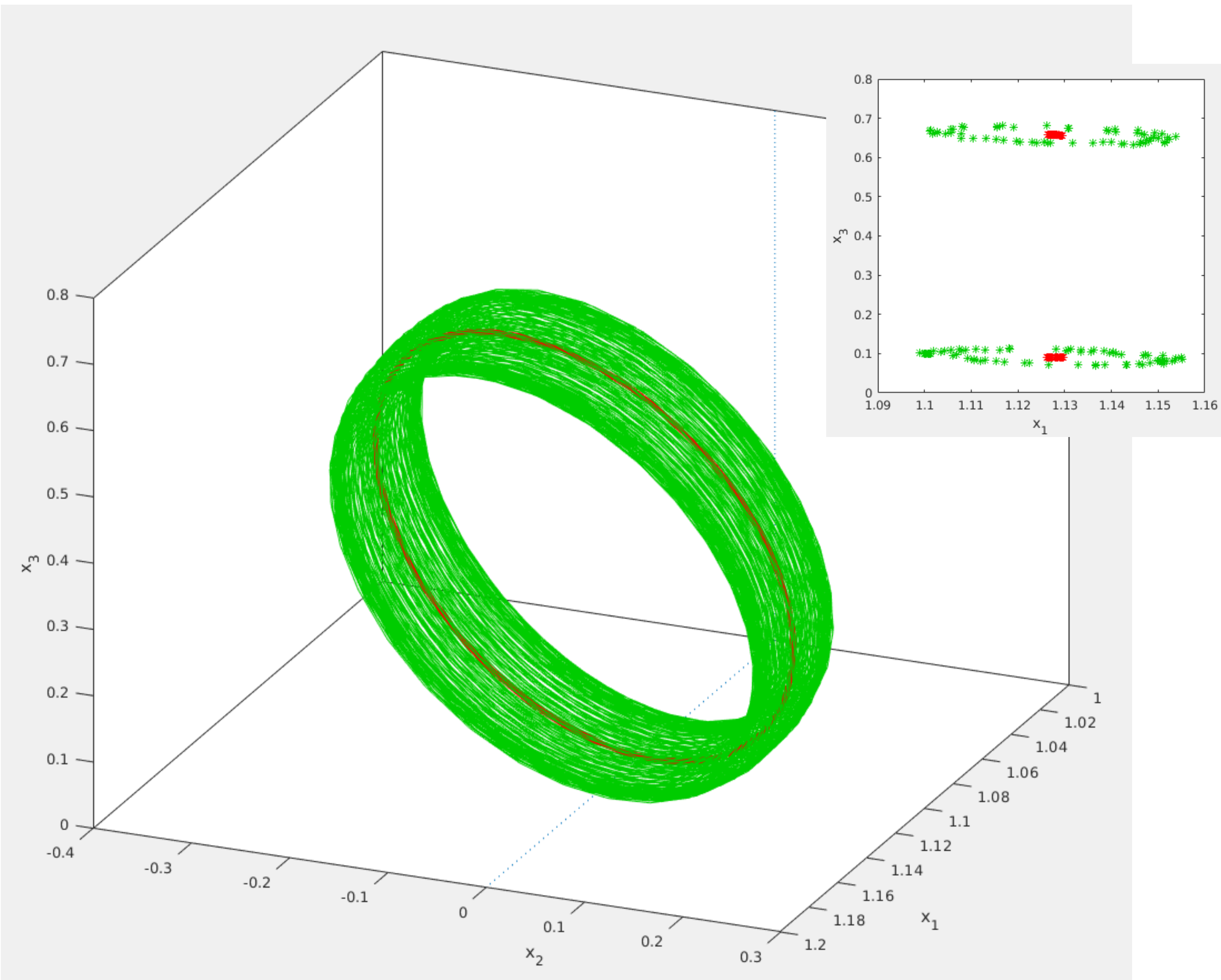
A 2D torus in a 3D system

The Lorenz 1984 system: A low-order model of atmospheric circulation [1]

$$\frac{dx}{dt} = -y^2 - z^2 - ax + aF,$$
$$\frac{dy}{dt} = xy - bxz - y + G,$$
$$\frac{dz}{dt} = bxy + xz - z,$$

where x represents the strength of the globally averaged westerly current and y and z and are the strength of the cosine and sine phases of a chain of superposed waves.

Parameters: a = 0.25, b = 4, F = 2, G = 1.67



[1] E. N. Lorenz, Irregularity: a fundamental property of the atmosphere, *Tellus A*, 36, 98–110, 1984.