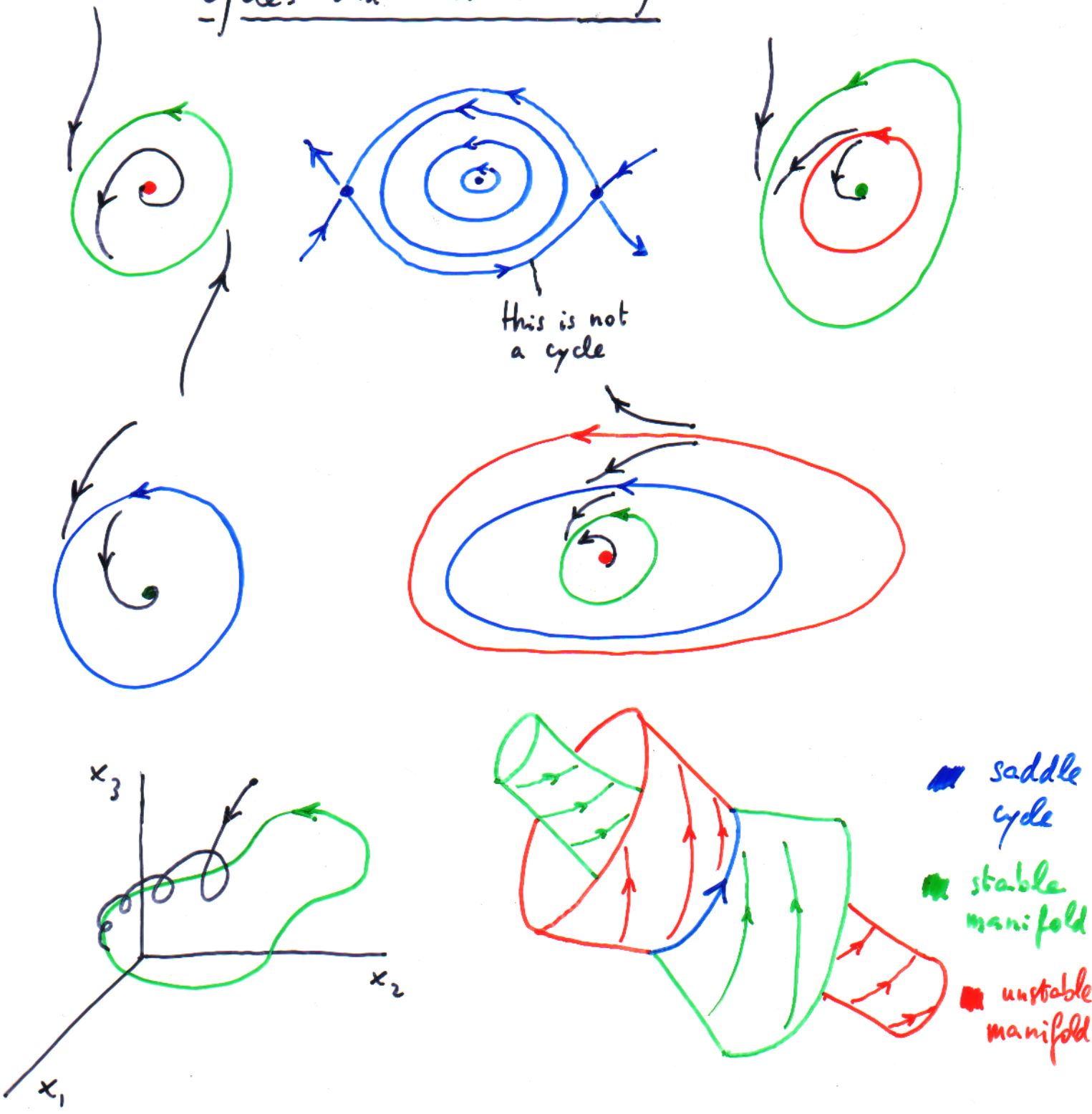


Cycles and their stability



Definition 1 (asymptotic stability of a cycle)

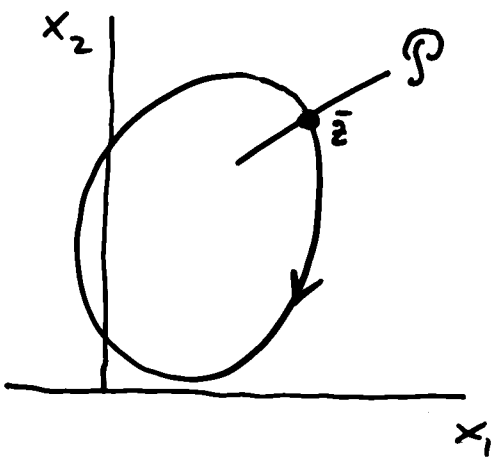
A cycle is (asymptotically) stable if a tube can be constructed around it such that all trajectories starting inside the tube remain in it and tend for $t \rightarrow \infty$ to the cycle.

Similarly for stability and instability

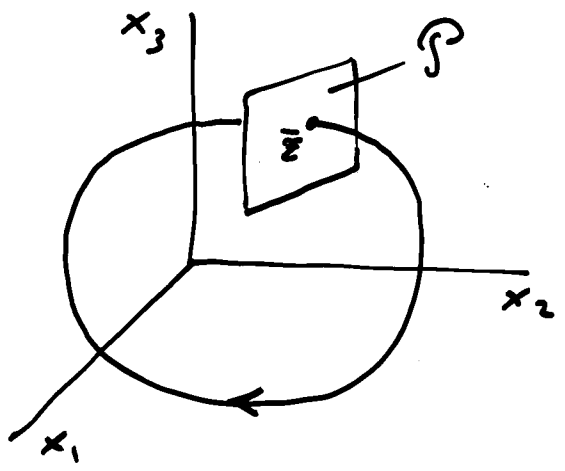
Definition 2 (simple stability)

A stable cycle is simply (critically) stable if it is not asympt. stable.

Poincaré section and map



$$\dot{x} = f(x)$$



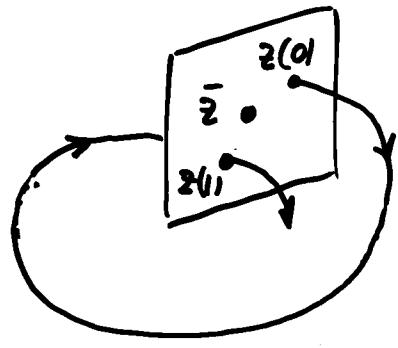
The Poincaré section \mathcal{P} is transversal to the cycle.

Nearby trajectories (in the tube) cross \mathcal{P} .

The cycle intersects \mathcal{P} at \bar{z} .

Consider a nearby trajectory starting from a point $z(0) \in \mathcal{P}$

The trajectory will return to \mathcal{P} , close to \bar{z} , at point $z(1)$ and then at points $z(2), z(3), \dots$



Obviously the cycle is asymptotically stable if and only if $z(t) \rightarrow \bar{z}$ for $t \rightarrow \infty$ for all $z(0)$ close to \bar{z}

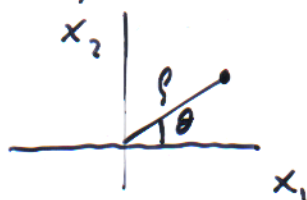
$$z(t+1) = P(z(t)) (*) \quad \text{with} \quad \bar{z} = P(\bar{z})$$

↑ Poincaré map (implicitly defined by $\dot{x} = f(x)$)

Theorem 1 The cycle is asymptotically stable if and only if \bar{z} is an asymptotically stable equilibrium of $(*)$.

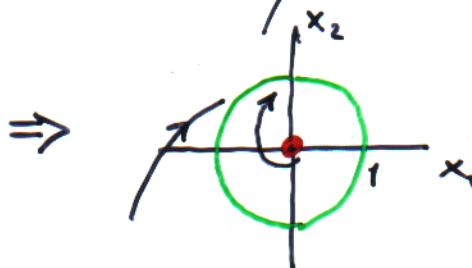
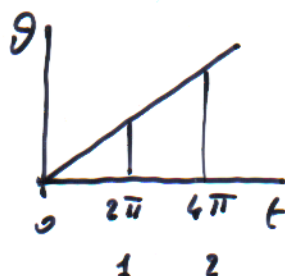
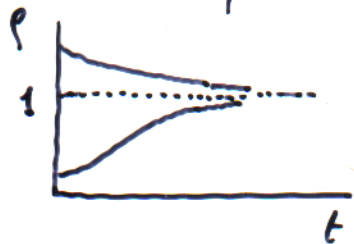
Ex. 1 The simplest limit cycle

$$\begin{cases} \dot{\rho} = \rho(1-\rho) \\ \dot{\theta} = 1 \end{cases}$$



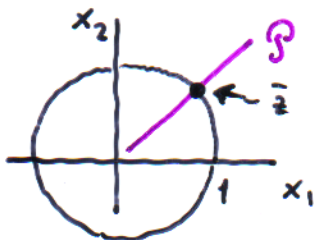
polar coordinates

The two equations are decoupled \Rightarrow easy



Let us see how we can proceed with Theorem 1.

(i) Fix a Poincaré section \mathcal{P} , for example, the straight line $x_2 = x_1$



(ii) Fix a coordinate system z on \mathcal{P} , for example,



$$z = \frac{1}{\rho}$$

(iii) Derive the Poincaré map $z(t+1) = P(z(t))$ $t = \text{integer}$

For this let $y(t) = \frac{1}{\rho(t)}$, $t = \text{real}$. Hence, $\dot{y} = -\frac{1}{\rho^2} \dot{\rho}$, i.e.

$$\dot{y} = -\frac{1}{\rho^2} \rho(1-\rho) = -y + 1 \Rightarrow y(t) = e^{-t} y(0) + 1 - e^{-t}$$

Since $y(0) = z(0)$, $y(2\pi) = z(1)$, $y(4\pi) = z(2)$, ...

we obtain

$$z(t+1) = e^{-2\pi} z(t) + 1 - e^{-2\pi} \quad (*)$$

(iv) The Poincaré map (*) is a first-order discrete-time linear system. Its eigenvalue is $e^{-2\pi} < 1$. Hence (*) is asymptotically stable, i.e. $z(t) \rightarrow \bar{z}$

Linearization of the Poincaré map

On the Poincaré section \mathcal{P} we have

$$z(t+1) = P(z(t)) \quad (*)$$

with $\bar{z} = P(\bar{z})$.

The stability of \bar{z} in (*) can be studied through linearization

$$\delta z(t) = z(t) - \bar{z} \quad \delta z(t+1) = \left. \frac{\partial P}{\partial z} \right|_{\bar{z}} \delta z(t)$$

$\left. \frac{\partial P}{\partial z} \right|_{\bar{z}}$ is the Jacobian matrix of the Poincaré map

$\left. \frac{\partial P}{\partial z} \right|_{\bar{z}}$ has $(n-1)$ eigenvalues because z is $(n-1)$ -dimensional

Theorem 2 (linearization method + Thm. 1)

Let μ_i , $i = 1, \dots, n-1$ be the eigenvalues of $\left. \frac{\partial P}{\partial z} \right|_{\bar{z}}$

Then $|\mu_i| < 1 \quad \forall i \Rightarrow$ the cycle is asymptotically stable

Remark 1 The eigenvalues μ_1, \dots, μ_{n-1} are sometimes called Floquet's coefficients

Theorem 3 If one (or more) $|\mu_i| > 1$ the cycle is unstable

Remark 2 If $|\mu_i| \leq 1$ and at least one $|\mu_i| = 1$ nothing can be said on the stability of the cycle

Second order systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

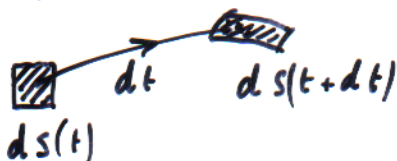
special properties hold for these systems

Theorem 4 (Bendixon) (non existence condition)

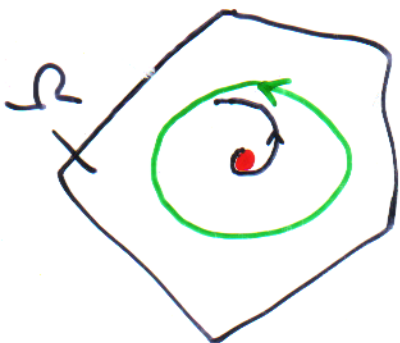
If the divergence $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$ does not change sign in a domain $\Omega \subset \mathbb{R}^2$ (and, at most, annihilates on a line) then there are no cycles in Ω .

Proof

$$(dS)^\cdot = (\text{div} f) \cdot dS$$



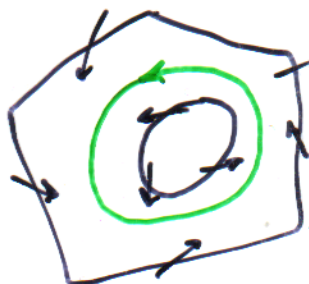
Thus, $\text{div} f > 0 \Rightarrow$ trajectories diverge
 $\text{div} f < 0 \Rightarrow$ trajectories converge



If there exists a stable cycle with an unstable equilibrium inside, trajectories must diverge (close to the equilibrium) and converge (close to the limit cycle).

Theorem 5 (Poincaré) (existence condition)

If there are no equilibria in an annular domain $A \subset \mathbb{R}^2$ and trajectories enter in A from its boundary, then there is at least one cycle in A .



Proof. Since trajectories entering in A cannot cross one each other and cannot tend toward an equilibrium, they must converge to a cycle.

Remark 3 Theorem 5 holds also if trajectories leave A .

Equilibria and cycles

6

Consider a second-order system

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

and assume that all its equilibria are hyperbolic (i.e. all Jacobian $\left. \frac{\partial f}{\partial x} \right|_{\bar{x}}$ have eigenvalues with non zero real parts)

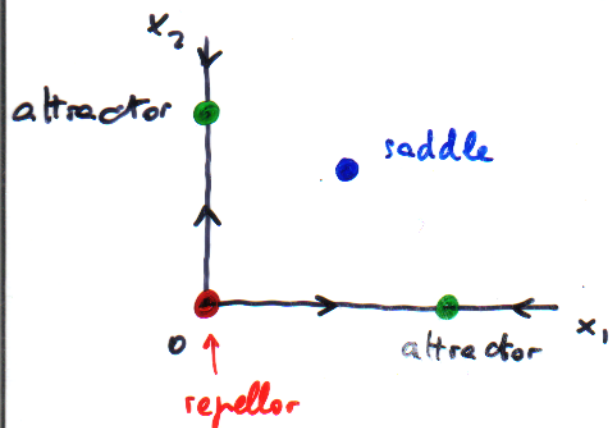
Then, the following property holds.

Theorem 6 (Poincaré)

Inside a cycle there must be an odd number of equilibria. More precisely, if s is the number of saddles, there must be $(s+1)$ equilibria which are not saddles.

Ex. 2 Bacteria competition (see Ex. 7 Lecture 3)

$$\begin{cases} \dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ \dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_1 x_2 \end{cases}$$



The system is positive (trajectories remain in the 1st quadrant)

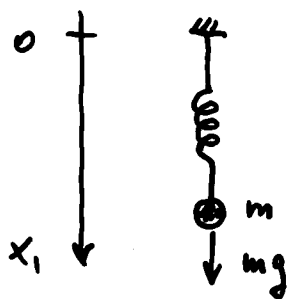
Cycles cannot exist in the first quadrant, because they would have inside either no equilibria or one saddle.

Problems

7

(2)

P. 1 - Consider the following mechanical system and assume that the spring is "linear" (i.e. its force is proportional to x_1) and that a viscous friction acts on the mass (i.e. a force opposed to the motion and proportional to the speed x_2 of the mass)



Write the state equations and prove that in such a system cycles cannot exist.

P. 2 ⁽¹⁾

Derive the state equations $\dot{x}_1 = \dots$ $\dot{x}_2 = \dots$ of the model described in polar coordinates in Ex. 1.

P. 3 ⁽¹⁾

Say why cycles cannot exist in the epidemic model

$$\dot{x}_1 = -\alpha x_1 x_2$$

$x_1 =$ susceptible

$$\dot{x}_2 = \alpha x_1 x_2 - \beta x_2$$

$x_2 =$ infective

P. 4 ⁽³⁾

Consider the prey-predator model described in problem P.1 of Lecture 4, and prove, by means of Theorem 5 (actually a slight modification of Thm. 5), that under conditions (1), (2) there exists a cycle (Hint: define the annular domain A by using segments of the x_1 and x_2 axis for the external boundary and a small circle around $\bar{x}^{(1)}$ for the internal boundary).