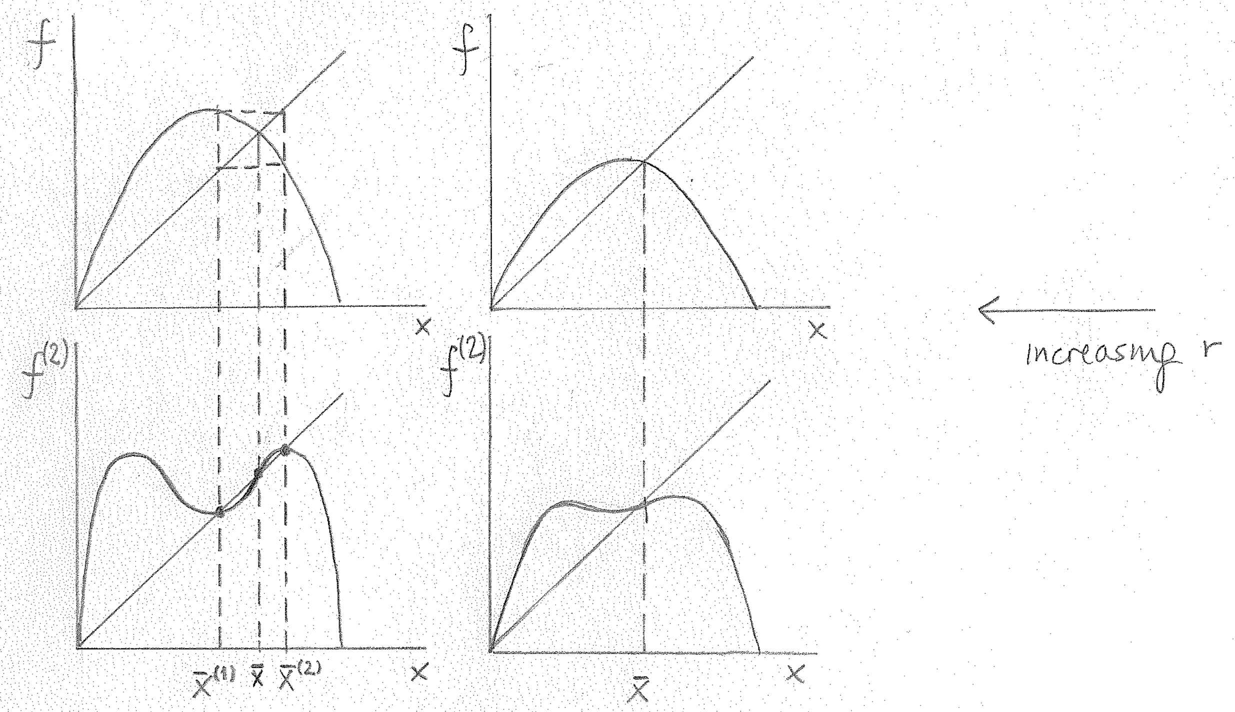
Cycles and their stability in discrete time

A period-T cycle $\gamma = \{\bar{x}^{(1)}, \bar{x}^{(2)}, \dots, \bar{x}^{(T)}\}$ in discrete time corresponds to T equilibria $\bar{x}^{(1)}, \bar{x}^{(2)}, \dots, \bar{x}^{(T)}$ of the <u>iterated map</u>

$$f^{(T)}(x) = \underbrace{f(f(\cdots f(x)\cdots))}_{T\text{-times}}$$

Ex: quadratic map



Note: the equilibria $\bar{x}^{(1)}, \bar{x}^{(2)}, \dots, \bar{x}^{(T)}$ of $f^{(T)}$ have the same stability properties, e.g. they have the same associated eigenvalues

$$J^{(T)}(x) = J(f^{(T-1)}(x)) \cdot J(f^{(T-2)}(x)) \cdot \dots \cdot J(f(x)) \cdot J(x)$$

$$J^{(T)}(\bar{x}^{(1)}) = J(\bar{x}^{(T)}) \cdot J(\bar{x}^{(T-1)}) \cdot \dots \cdot J(\bar{x}^{(2)}) \cdot J(\bar{x}^{(1)})$$

$$J^{(T)}(\bar{x}^{(2)}) = J(\bar{x}^{(1)}) \cdot J(\bar{x}^{(T)}) \cdot J(\bar{x}^{(T-1)}) \cdot \dots \cdot J(\bar{x}^{(2)})$$

$$\vdots$$

$$J^{(T)}(\bar{x}^{(K)}) = J(\bar{x}^{(K-1)}) \cdot \dots \cdot J(\bar{x}^{(1)}) \cdot J(\bar{x}^{(T)}) \cdot J(\bar{x}^{(T-1)}) \cdot \dots \cdot J(\bar{x}^{(K)})$$

Result: stability of γ $\equiv$ stability of any of the equilibria $\bar{x}^{(K)}$ of $f^{(T)}$

Definitions of stability for cycles in continuous time

Definition 1 : (local) stability

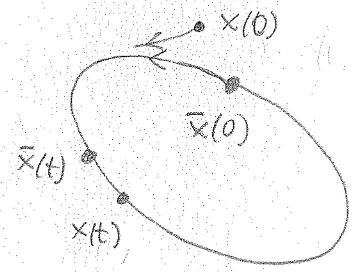
A cycle γ is (locally) stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that all trajectories starting in a δ-tube around γ remain in an ε-tube around γ for all $t > 0$.

Definition 2: asymptotic stability

A stable cycle γ is asymptotically stable if all perturbed trajectories $x(t)$ (starting in the δ-tube) tend to γ for $t \to +\infty$

In words : any small perturbation is asymptotically absorbed

Note : given $\bar{x}(0) \in \gamma$ and a perturbed initial condition $x(0)$ close to γ, it is not true that $\delta x(t) = x(t) - \bar{x}(t) \longrightarrow 0$.
$x(t) \to \gamma$, but the perturbation along the cycle is not absorbed



Definition 3 : (local) instability

A cycle which is not stable is called unstable

Definition 4 : basin of attraction

Given an asymptotically stable cycle γ, the set

$$B(\gamma) = \{ x(0) : x(t) \longrightarrow \gamma \}$$

is called basin of attraction of γ.

Definition 5 : global stability

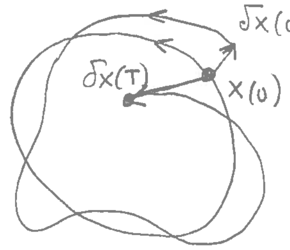
If $B(\gamma)$ coincides with $\mathbb{R}^n$ (with the exception of a set of zero measure) γ is called globally stable.

Numerical computation of the Floquet multipliers (shooting)

Idea: instead of constructing the (nonlinear) Poincaré map and then linearize it, we linearize the continuous-time dynamics around the cycle γ (variational dynamics)

$$\dot{x}(t) = f(x(t)) \quad , \quad x(0) \in \gamma$$

$$\delta\dot{x}(t) = J(x(t)) \cdot \delta x(t)$$



After one period, the initial perturbation $\delta x(0)$ becomes $\delta x(T)$ (at 1-st order)

Note on the superposition principle: denoting with $m^{(k)}$ the perturbation $\delta x(T)$ obtained with $\delta x(0) = e^{(k)} = [0 \cdots 0\,1\,0 \cdots 0]^T$, $k = 1, \dots, n$, then for any $\delta x(0)$ we have

k-th position

$$\delta x(T) = M\,\delta x(0) \quad \text{with } M = [m^{(1)}, m^{(2)}, \dots, m^{(n)}]$$

M is called monodromy matrix associated to $x(0) \in \gamma$

It always has a (trivial) eigenvalue $= 1$ with eigenvector $f(x(0))$

Result: the nontrivial eigenvalues of M coincide with those of any linearized Poincaré map.

Property: $\det M > 0$