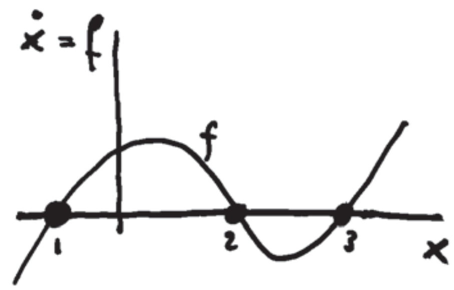
Equilibria and isoclines

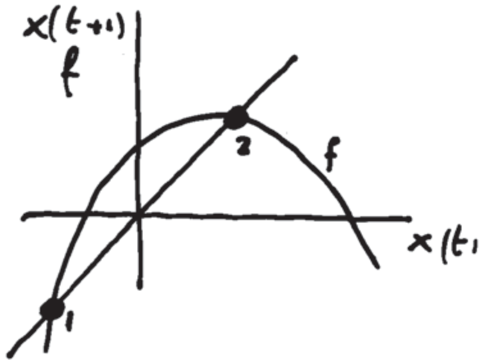
Definition 1 (equilibrium)

An equilibrium is a state $\bar{x}$ such that $x(0) = \bar{x}$ implies $x(t) = \bar{x} \quad \forall t \geq 0$.

Consequence In continuous-time systems $\dot{x} = f(x)$ equilibria $\bar{x}$ are such that $f(\bar{x}) = 0$, while in discrete-time systems $x(t+1) = f(x(t))$ equilibria $\bar{x}$ are such that $\bar{x} = f(\bar{x})$

Ex. 1 1-st order systems

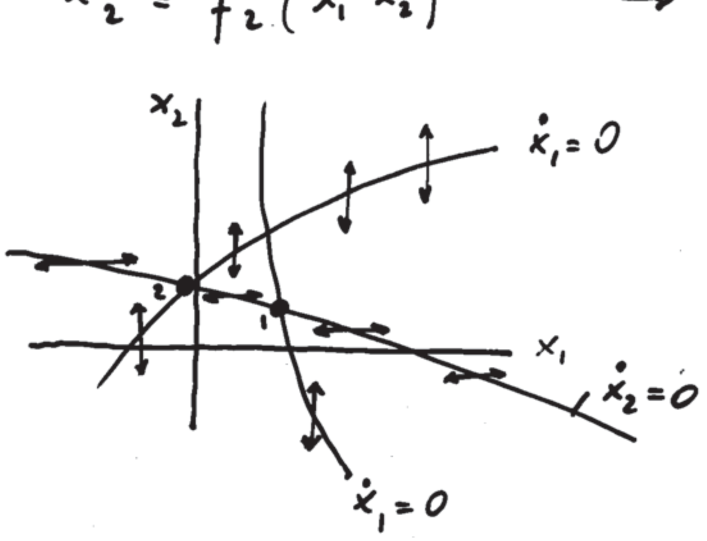


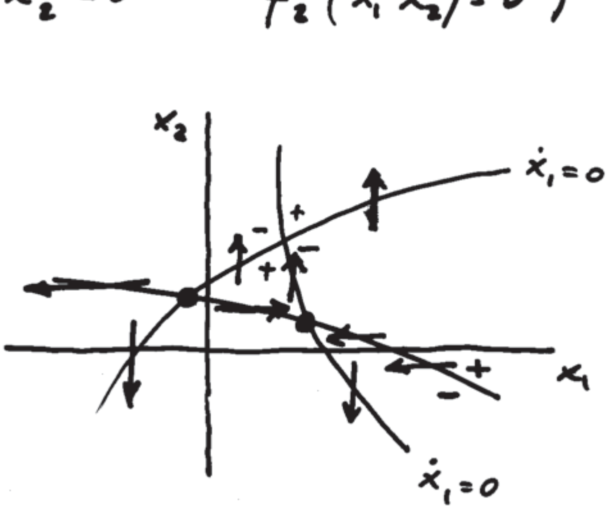


Ex. 2 2-nd order continuous time systems

$$\dot{x}_1 = f_1(x_1 \; x_2) \quad \Rightarrow \quad \dot{x}_1 = 0 \quad f_1(x_1 \; x_2) = 0$$
$$\dot{x}_2 = f_2(x_1 \; x_2) \quad \Rightarrow \quad \dot{x}_2 = 0 \quad f_2(x_1 \; x_2) = 0$$

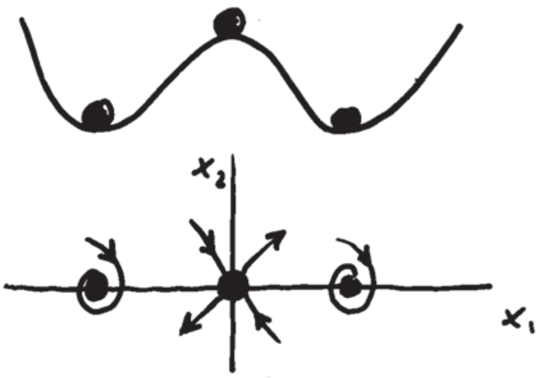
(null) isoclines

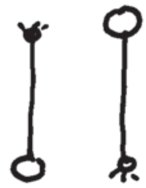


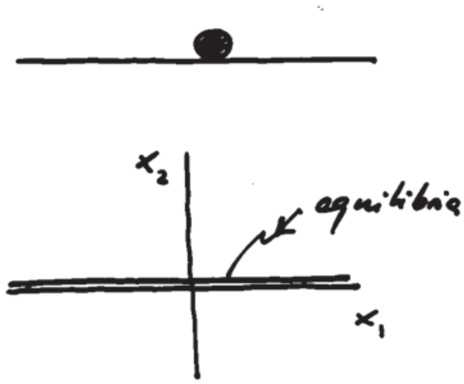


Multiplicity of equilibria

We have already seen that equilibria can be multiple







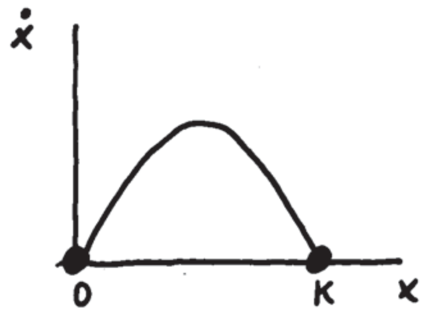
Ex. 3. Logistic growth

$x(t)$ = biomass at time t

$$\left.\begin{aligned} \dot{x} &= n(x)\cdot x - m(x)x \\ n(x) &= n_0 - \alpha x \\ m(x) &= m_0 + \beta x \end{aligned}\right\} \Rightarrow \quad \dot{x} = rx\left(1 - \frac{x}{K}\right)$$

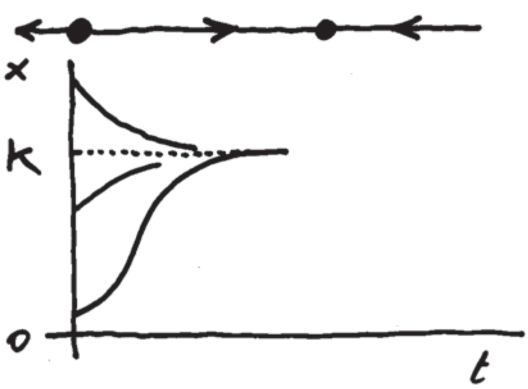
$r = n_0 - m_0$

$K = \dfrac{n_0 - m_0}{\alpha + \beta}$



two equilibria $\bar{x} = 0$

$\bar{x} = K$ carrying capacity

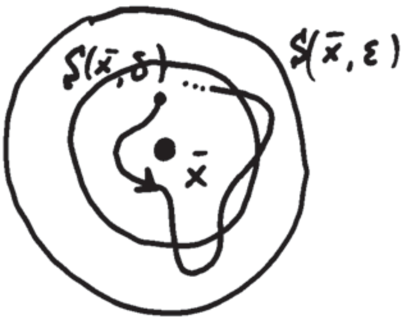


the population biomass tends toward its carrying capacity

Stability

Definition 2 ((local) stability) (Liapunov 1892)
An equilibrium $\bar{x}$ is (locally) stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\| x(0) - \bar{x} \| < \delta \Rightarrow \| x(t) - \bar{x} \| < \varepsilon \qquad \forall\, x(0),\ t > 0$$



There is no small perturbation of the state, after which the system moves far away from the equilibrium

Definition 3 (asymptotic stability)
A stable equilibrium $\bar{x}$ is asymptotically stable if all perturbed trajectories $x(t)$ tend to $\bar{x}$ for $t \to \infty$

In words : any small perturbation is asymptotically absorbed

Definition 4 ((local) instability)
An equilibrium which is not stable is called unstable.

Definition 5 (basin of attraction)
Given an asymptotically stable equilibrium $\bar{x}$ the set

$$B(\bar{x}) = \{ x(0) : x(t) \to \bar{x} \}$$

is called basin of attraction

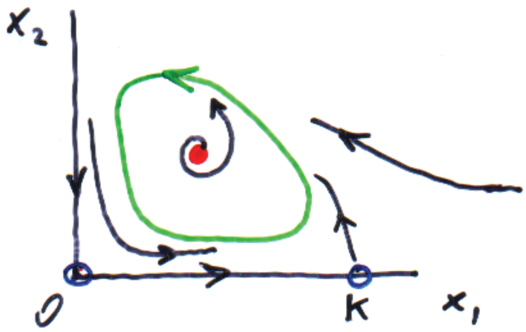
Definition 6 (global stability)
If $B(\bar{x})$ coincides with R^n (with the exception of a set of zero measure) $\bar{x}$ is called globally stable

Examples

Ex. 4 Logistic growth

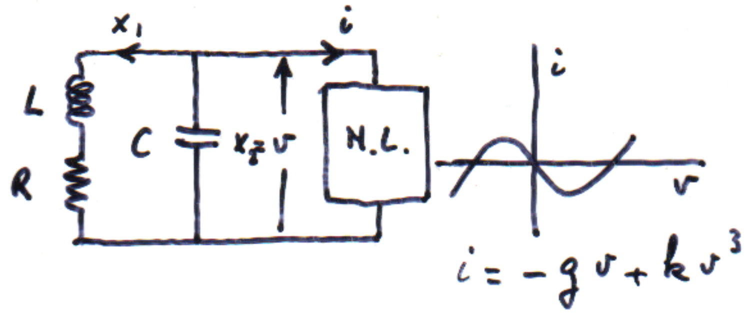
0 is unstable and K is stable
K is asymptotically stable
K is globally stable (in R_+)

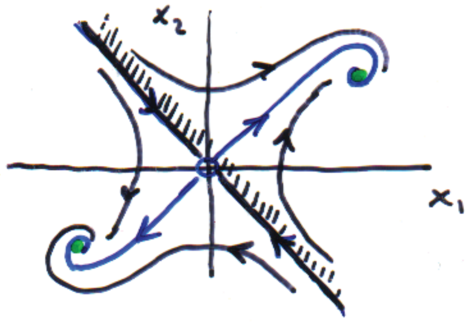
Ex. 5 Prey-predator model



There are three equilibria. They are all unstable (one is a focus and two are saddles)

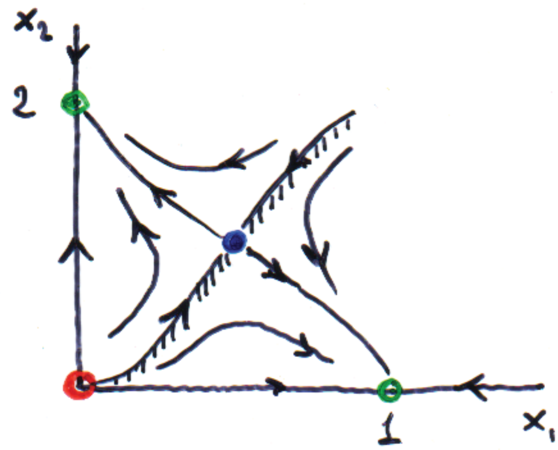
Ex. 6 Electric circuit (bistable)





Ex. 7 Bacteria competition

$$\dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2$$
$$\dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_1 x_2$$



4 equilibria : 2 stable, 2 unstable

control from 2 to 1 : antibiotic + yeast

Problems

P. 1.[1] Prove that a linear system $\dot{x} = Ax$ has a unique equilibrium (namely, the origin $\bar{x} = 0$) if A is non singular (i.e. if A does not have a null eigenvalue). What happens if A is singular? Formulate the analogous results for discrete-time linear systems.

P. 2.[1] Sketch the state portrait of the following mechanical system and indicate the basin of attraction of the two stable equilibria

P. 3.[1] Consider the following discrete-time 1-st order model

$$x(t+1) = \frac{1}{2 - x(t)}$$

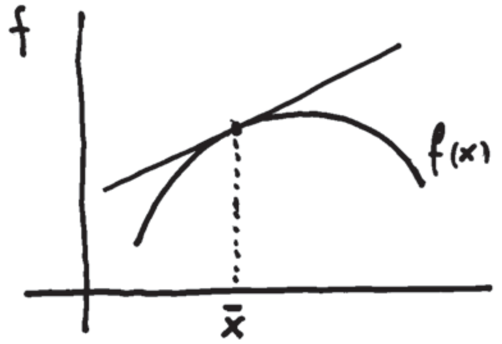
and show that this model has only one equilibrium, namely $\bar{x} = 1$. Then, use Moran's construction to prove that the equilibrium is unstable. Finally, observe that all trajectories tend toward $\bar{x}$, even if $\bar{x}$ is unstable. (This model describes the evolution of a genetic desease in a population: t is the generation and $x(t)$ is the probability that a randomly selected individual of generation t is sick).

P. 4.[2] Using the model described in Ex. 7 and the isoclines try to give a formal support to the state portrait shown in Ex. 7.

Local approximation

$$y = f(x) \qquad y = f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}}(x-\bar{x}) + \frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{\bar{x}}(x-\bar{x})^2 + \cdots$$

local approximation $\qquad y = f(\bar{x}) + \left.\dfrac{df}{dx}\right|_{\bar{x}}(x-\bar{x})$



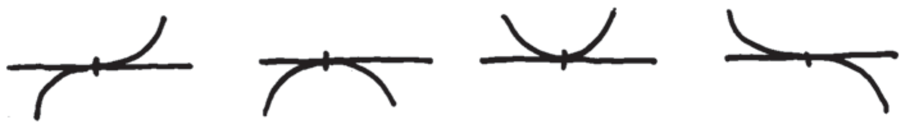
geometric interpretation of linearization

The sign of $\left.\dfrac{df}{dx}\right|_{\bar{x}}$ is important

$\left.\dfrac{df}{dx}\right|_{\bar{x}} > 0 \qquad f(\cdot)$ is locally increasing

$\left.\dfrac{df}{dx}\right|_{\bar{x}} < 0 \qquad f(\cdot)$ is locally decreasing

$\left.\dfrac{df}{dx}\right|_{\bar{x}} = 0 \qquad$ critical case : nothing can be said



Jacobian matrix (1)

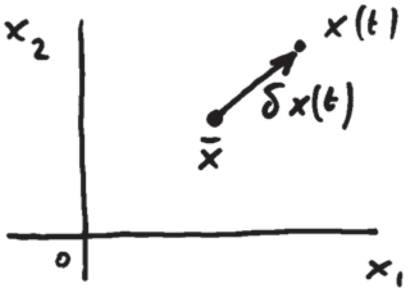
$$\dot{x}(t) = f\big(x(t)\big)$$

$$\bar{x} = \text{equilibrium } \big(f(\bar{x}) = 0\big)$$

Taylor's expansion

$$\dot{x} = f(\bar{x}) + \left[\frac{\partial f}{\partial x}\right]_{\bar{x}} (x - \bar{x}) + \underbrace{\cdots}_{O(x-\bar{x})^2}$$

$$\left[\frac{\partial f}{\partial x}\right] = \underline{\text{Jacobian}}\,(\text{matrix}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = J$$



$$\delta x(t) = x(t) - \bar{x}$$

$$\delta \dot{x}(t) = \big(\dot{x}(t) = \big) \left[\frac{\partial f}{\partial x}\right]_{\bar{x}} \delta x(t) + \cdots$$

Linearized system

$$\boxed{\delta \dot{x} = J\, \delta x}$$

Ex. 1 - Logistic growth

$$\dot{x} = r x \left(1 - \frac{x}{K}\right) \qquad \bar{x} = 0 \Rightarrow \delta\dot{x} = r x$$

$$\left(J = r - \frac{2 r x}{K}\right) \qquad \bar{x} = K \Rightarrow \delta\dot{x} = -r x$$

Jacobian matrix (2)

$$x(t+1) = f(x(t))$$

$$\bar{x} = \text{equilibrium} \quad (\bar{x} = f(\bar{x}))$$

$$x(t+1) = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} (x(t) - \bar{x}) + \cdots$$

$$x(t+1) = \bar{x} + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} (x(t) - \bar{x}) + \cdots$$

$$\delta x(t+1) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} \delta x(t) + \cdots \qquad \boxed{\delta x(t+1) = J\, \delta x(t)}$$

linearized system

Ex. 2 - Age structured population

$x_i(t)$ = # individuals of age i in year t

$$x_1(t+1) = x_3(t) \cdot F(x_3(t)) \qquad \leftarrow F = \text{fertility}$$

$$x_2(t+1) = s_2 x_1(t)$$

$$x_3(t+1) = s_3 x_2(t) + s^* x_3(t)$$

s = survival

mature individuals

$$\delta x(t+1) = J\, \delta x(t)$$

$(1 - s^*)/s_2 s_3$

$$J = \begin{vmatrix} 0 & 0 & F(\bar{x}_3) + \left.\frac{\partial F}{\partial x_3}\right|_{\bar{x}_3} \bar{x}_3 \\ s_2 & 0 & 0 \\ 0 & s_3 & s^* \end{vmatrix}$$

Linearisation method

$$\left.\begin{aligned} \dot{x} &= f(x) \\ x(t+1) &= f(x(t)) \end{aligned}\right\} \bar{x} = \text{equilibrium} \Rightarrow J = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}}$$

Theorem 1 J = asympt. stable $\Rightarrow$ $\bar{x}$ = asympt. stable

In words : if the linearised system is asymptotically stable, the equilibrium $\bar{x}$ is such

Theorem 2 J = exponentially unstable $\Rightarrow$ $\bar{x}$ = unstable

at least one eigenvalue of J is strictly unstable ($Re(\lambda_i) > 0$ or $|\lambda_i| > 1$)

Comment If J is simply stable or weakly unstable nothing can be said on the stability of $\bar{x}$.

Ex. 3. Logistic growth

$\dot{x} = r x \left(1 - \frac{x}{K}\right)$

$\bar{x} = 0 \quad J = r \Rightarrow \bar{x}$ = unstable

$\bar{x} = K \quad J = -r \Rightarrow \bar{x}$ = asympt. stable

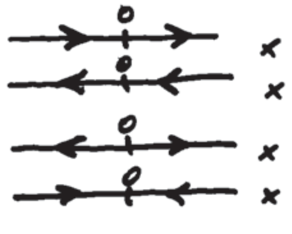
Ex. 4. Quadratic and cubic systems

$\dot{x} = x^2 \quad \bar{x} = 0 \quad J = 2x\big|_{\bar{x}} = 0 \Rightarrow ?$

$\dot{x} = -x^2 \quad \bar{x} = 0 \quad J = -2x\big|_{\bar{x}} = 0 \Rightarrow ?$

$\dot{x} = x^3 \quad \bar{x} = 0 \quad J = 0 \Rightarrow ?$

$\dot{x} = -x^3 \quad \bar{x} = 0 \quad J = 0 \Rightarrow ?$



Problems

P. 1.[3] Consider the prey - predator model

$$\begin{cases} \dot{x}_1 = r\, x_1 \left(1 - \dfrac{x_1}{K}\right) - \dfrac{a\, x_1}{b + x_1}\, x_2 & \text{prey} \\[2ex] \dot{x}_2 = e\, \dfrac{a\, x_1}{b + x_1}\, x_2 - d\, x_2 & \text{predator} \end{cases}$$

and assume that the positive parameters a, b, e, d, r, K satisfy the following inequalities

$$b < K \quad (1) \qquad\qquad \frac{ea}{d} > \frac{K+b}{K-b} \quad (2)$$

. Show that under conditions (1, 2) the system has two trivial equilibria

$$\bar{x}^{(1)} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad \text{and} \qquad \bar{x}^{(2)} = \begin{vmatrix} K \\ 0 \end{vmatrix}$$

and one strictly positive equilibrium

$$\bar{x}^{(3)} = \begin{vmatrix} \bar{x}_1 \\ \bar{x}_2 \end{vmatrix} \qquad \bar{x}_i > 0 \quad i = 1, 2$$

- Show, through linearization, that $\bar{x}^{(1)}$, $\bar{x}^{(2)}$ and $\bar{x}^{(3)}$ are unstable.
- Compare with Ex. 5 of Lecture 3.

P. 2[3] - Discuss, through linearization, the stability of the four equilibria shown in Ex. 7 of Lecture 3 (pay attention to inequalities among parameters).

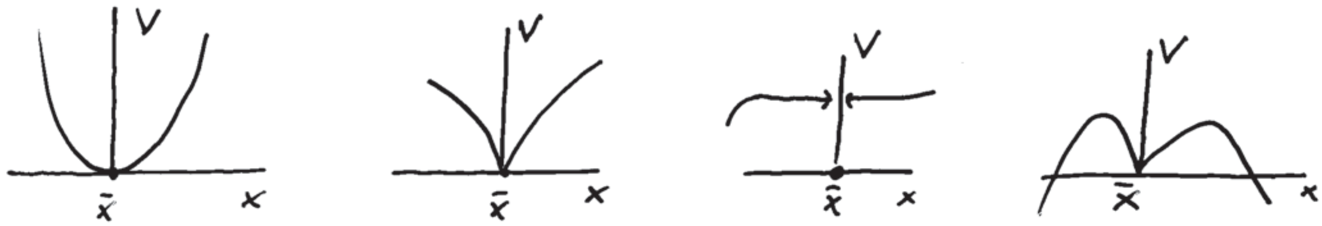
POSITIVE DEFINITE FUNCTIONS

$x \in \mathbb{R}^n \qquad V(x) : \mathbb{R}^n \to R$

__Definition__ $V(\cdot)$ is positive definite at $\bar{x}$ iff

(a) $V(\bar{x}) = 0$

(b) $V(x) > 0 \quad \forall x \neq \bar{x}$ in a neighborhood of $\bar{x}$



In $\mathbb{R}^2$

$V(x) = x_1^2 + x_2^2 \qquad V(x) = x_1 - \log x_1 + x_2 - \log x_2 - 2$

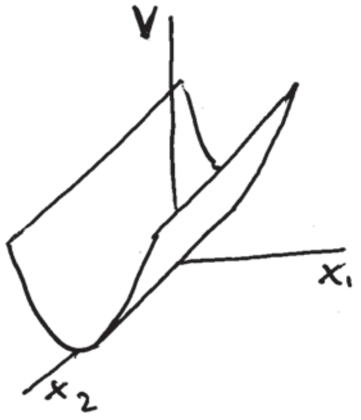
$V(x) = a x_1^2 + 2 b x_1 x_2 + c x_2^2 \quad$ with $a > 0$, $ac > b^2$

__Definition__ $V(\cdot)$ is positive semidefinite at $\bar{x}$ iff

(a) $V(\bar{x}) = 0$

(b) $V(x) \geq 0 \quad \forall x \neq \bar{x}$ in a neighborhood of $\bar{x}$

(c) $V(x) = 0$ for some x close to $\bar{x}$



$V = x_1^2$ is posit. semid. in $\mathbb{R}^2$

__Definition__ $V(\cdot)$ is negative (semi)definite iff $-V(\cdot)$ is positive (semi)definite

LIAPUNOV METHOD [1892]

$$\dot{x} = f(x) \qquad f(\bar{x}) = 0$$

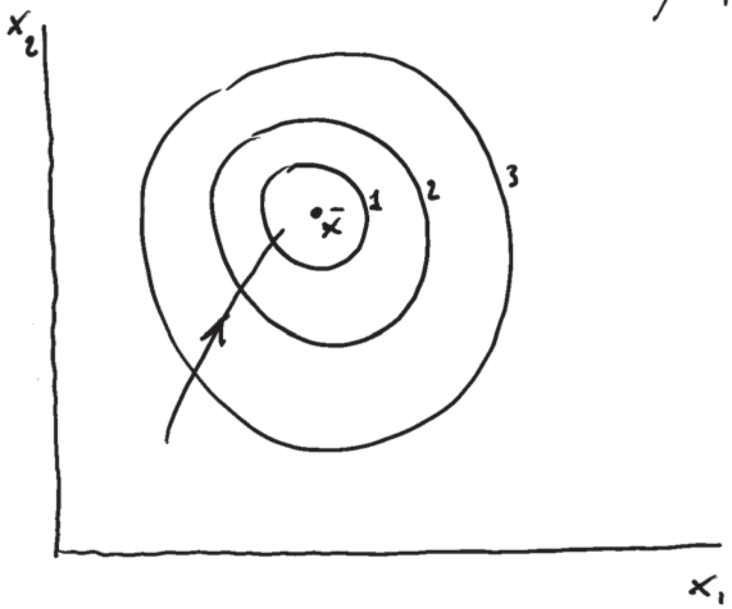
(a) $V(\cdot)$ posit. def. at $\bar{x}$

(b) $V(\cdot)$ regular (continuous with continuous derivatives)

(c) $\dot{V}(x) = \frac{\partial V}{\partial x} f = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \cdots + \frac{\partial V}{\partial x_n} f_n$ negative definite at $\bar{x}$

$$\Downarrow$$

$$\bar{x} = \text{asympt. stable}$$



(a) + (b) $\Rightarrow$ the lines at $V = \text{const.}$ are closed and ordered close to $\bar{x}$.

$\dot{V}$ neg. def. $\Rightarrow$ trajectories cross the lines $V = \text{const.}$ transversally from outside toward inside

<u>Remark</u> A function $V(\cdot)$ satisfying conditions (a), (b) and (c) is called a Liapunov function

<u>Krasowskii's method</u>

(a) + (b) + $\dot{V}$ negative semidefinite + Krasowskii condition (*)

$$\Downarrow$$

$$\bar{x} = \text{asympt. stable}$$

$$K = \{ x : \dot{V}(x) = 0 \}$$

(*) K does not contain trajectories close to $\bar{x}$ but $\neq \bar{x}$

LA SALLE METHOD

$\dot{x} = f(x) \qquad f(\bar{x}) = 0$

$V(\cdot)$ is a Liapunov (or Kresowskii) function

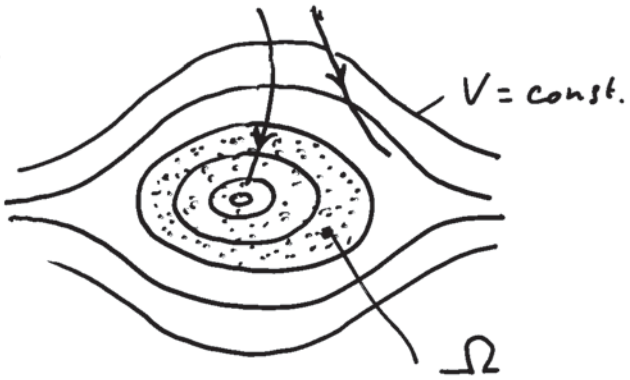
$$\Downarrow$$

$\bar{x}$ = asympt. stable

$$\Downarrow$$

$B(\bar{x}) = ?$

La Salle : $B(\bar{x}) \supset \Omega$ = the largest region contained in a closed line $V(x) = \text{const.}$ in which Liapunov (or Krasows.) conditions hold



Remark Ω is an estimate (from below) of $B(\bar{x})$

Remark quadratic positive definite functions $V(\cdot)$ have always closed lines $V = \text{const.}$

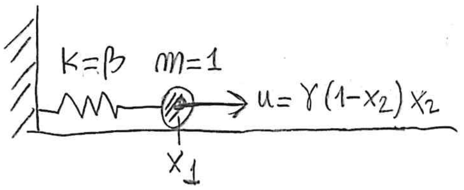
EXAMPLE (van der Pol equation)

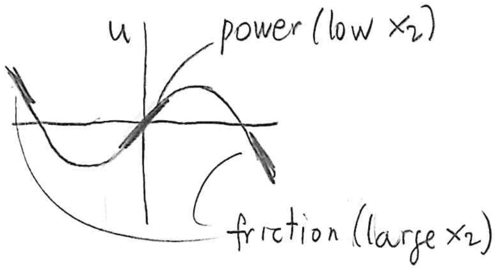
$$\dot{x}_1 = \alpha x_2$$

$$\dot{x}_2 = -\beta x_1 + \gamma(1 - x_2^2)\, x_2$$

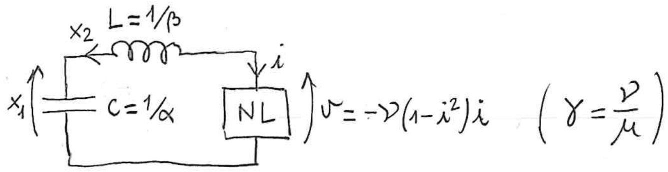
Mechanical interpretation

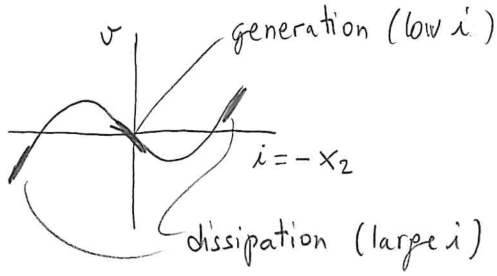
$(\alpha = 1)$





Electrical interpretation





Equilibrium $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$J = \begin{bmatrix} 0 & \alpha \\ -\beta & \gamma \end{bmatrix} \qquad \mathrm{tr}\, J = \gamma, \quad \det J = \alpha\beta \qquad \bar{x} \text{ as. stab. for } \begin{cases} \gamma < 0 \\ \alpha\beta > 0 \end{cases}$$

Assume $\alpha > 0$, $\beta > 0$, $\gamma < 0$ (note: the Van der Pol oscillator has $\gamma > 0$!)

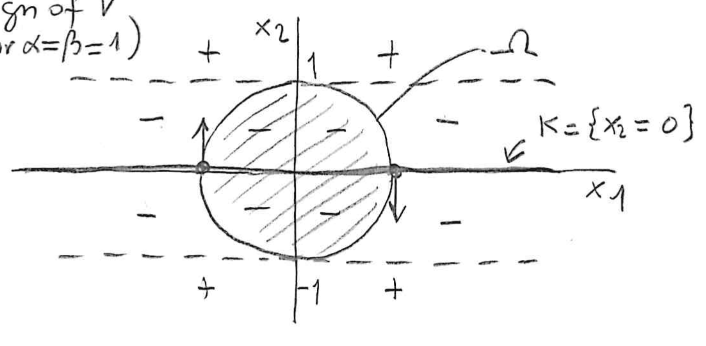
$V(x) = \frac{c_1}{2} x_1^2 + \frac{c_2}{2} x_2^2$, energy-like function with elliptic contour lines coefficient c_1 and c_2 to be determined to simplify the expression of $\dot{V}$

$$\dot{V} = c_1 x_1 \dot{x}_1 + c_2 x_2 \dot{x}_2 = \underset{\substack{\| \\ \beta}}{c_1} \cancel{x_1 \alpha x_2} + \underset{\substack{\| \\ \alpha}}{c_2} x_2 \left(-\cancel{\beta x_1} + \gamma(1 - x_2^2)\, x_2\right)$$

$$= \alpha\gamma x_2^2 (1 - x_2^2) \leq 0 \text{ for } |x_2| < 1$$

$$= 0 \text{ for } x_2 = 0 \text{ and } x_2 = \pm 1$$

sign of $\dot{V}$ (for $\alpha = \beta = 1$)



$\Rightarrow \bar{x}$ as. stab