Modello risorse - consumatori / prede - predatore di Rosenzweig - MacArthur

x_1 = densità/biomassa di risorse/prede

x_2 = " consumatori/predatori

$$\dot{x}_1 = r x_1 \left(1 - \frac{x_1}{k}\right) - \frac{a x_1}{b + x_1} x_2$$

$r x_1 \left(1 - \frac{x_1}{k}\right)$ → crescita logistica

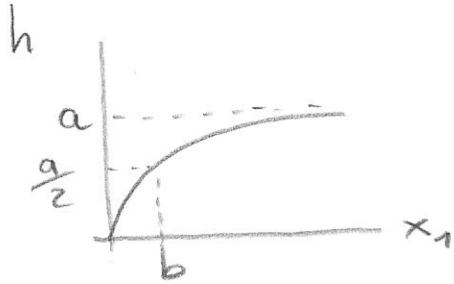
$\frac{a x_1}{b + x_1} x_2$ → mortalità per predazione

$$h(x_1) = \frac{a x_1}{b + x_1}$$

→ Risposta funzionale di Holling di tipo II

r = tasso intrinseco di crescita

k = capacità portante



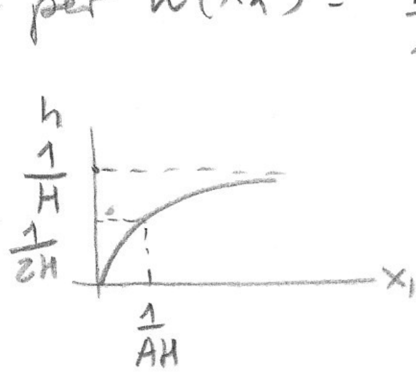
a = massima capacità predatoria (è pari al max di h)

b = costante di semi-saturazione (è il valore di x_1 per il quale h è metà del suo valore massimo)

NOTA: Altra formulazione per $h(x_1) = \dfrac{A x_1}{1 + A H x_1}$

A = attack rate

H = handling time



$$\Rightarrow \quad a = \frac{1}{H} \qquad b = \frac{1}{AH} \quad \longrightarrow \quad A = \frac{a}{b} \qquad H = \frac{1}{a}$$

$$\dot{x}_2 = e\,\frac{a x_1}{b+x_1}\,x_2 - m\,x_2$$

crescita per predazione

mortalità naturale

m = tasso di mortalità

e = efficienza di conversione

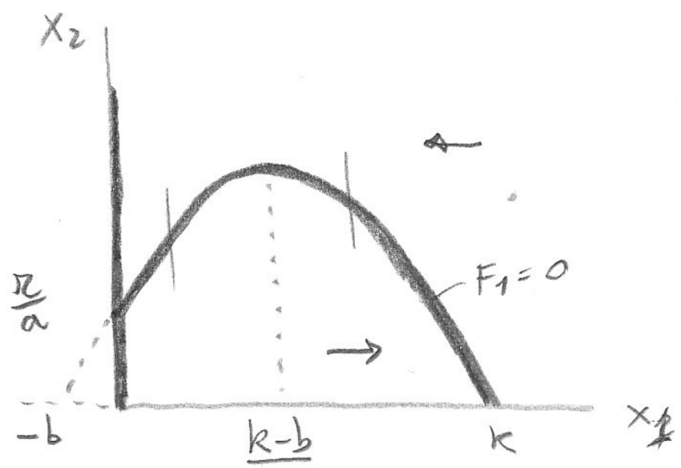
$$\boxed{\begin{aligned}\dot{x}_1 &= r x_1\left(1-\frac{x_1}{k}\right) - a\,\frac{x_1}{b+x_1}\,x_2 \\ \dot{x}_2 &= e\,\frac{a x_1}{b+x_1}\,x_2 - m x_2\end{aligned}}$$

Isocline - equilibri - stabilità - traiettorie

$$\dot{x}_1 = x_1\left[r\left(1-\frac{x_1}{k}\right) - \frac{a x_2}{b+x_1}\right] = x_1 \cdot F_1(x_1, x_2)$$

$$\dot{x}_2 = x_2\left[e\,\frac{a x_1}{b+x_1} - m\right] = x_2 \cdot F_2(x_1)$$

Isocline $\quad \dot{x}_1 = 0 \; \begin{cases} x_1 = 0 \\ x_2 = \frac{r}{a}\left(1-\frac{x_1}{k}\right)(b+x_1) \quad \leadsto F_1(x_1,x_2)=0 \end{cases}$



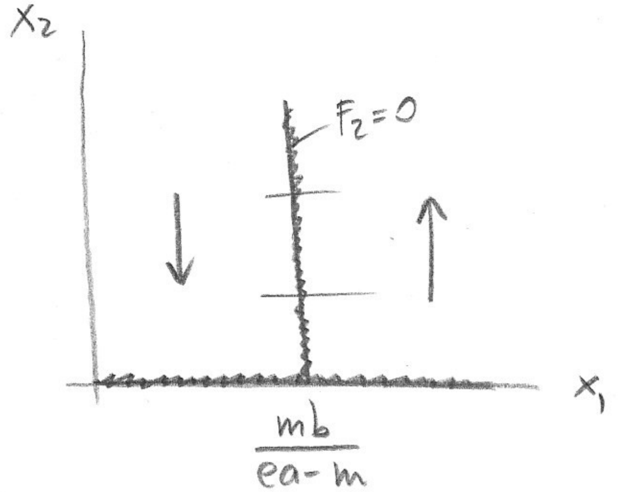
$(k > b)$

Isocline $\dot{x}_2 = 0$ — $x_2 = 0$

— $x_1 = \dfrac{mb}{ea-m} \rightsquigarrow F_2(x_1)=0$ $\quad ea - m > 0$

$\downarrow$

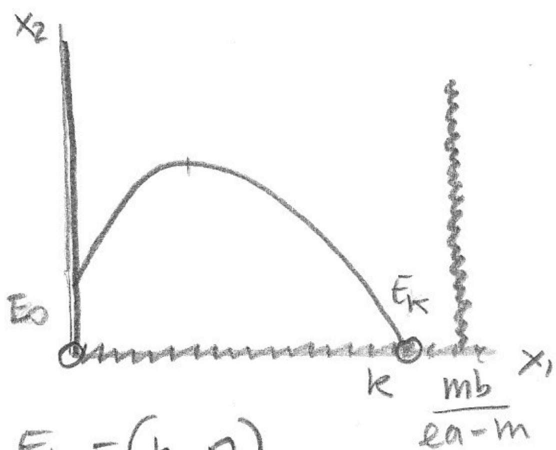
Se la risorsa x_1 è infinita, il consumatore è in grado di crescere ($\dot{x}_2 > 0$)



Equilibri $\longrightarrow$ $\cap$ isocline $\dot{x}_1 = 0$ e $\dot{x}_2 = 0$

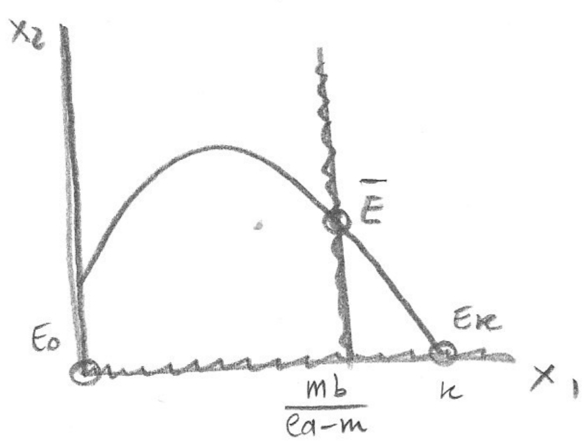
$\hookrightarrow x_i \geq 0$

- $k < \dfrac{mb}{ea-m}$



$E_0 = (0,0) \qquad E_k = (k, 0)$

- $k > \dfrac{mb}{ea-m}$



$E_0 = (0,0) \qquad E_k = (k, 0) \qquad \bar{E} = (\bar{x}_1, \bar{x}_2) \,/\, F_1(\bar{x}_1, \bar{x}_2) = 0$

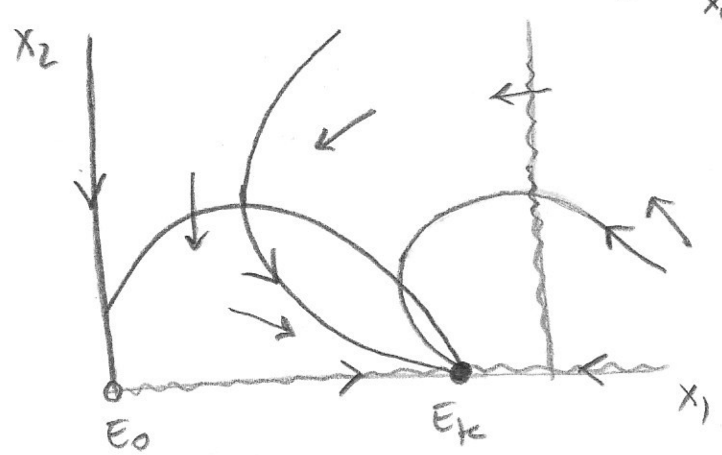
NOTA : $\frac{mb}{ea-m} > \frac{k-b}{2} \Rightarrow \bar{E}$ è a dx del max della parabola

$\frac{mb}{ea-m} < \frac{k-b}{2} \Rightarrow \bar{E}$ è a sx " "

NOTA : Gli assi sono invarianti
$x_i(0)=0 \rightarrow \dot{x}_i(0)=0 \rightarrow x_i(t)=0$

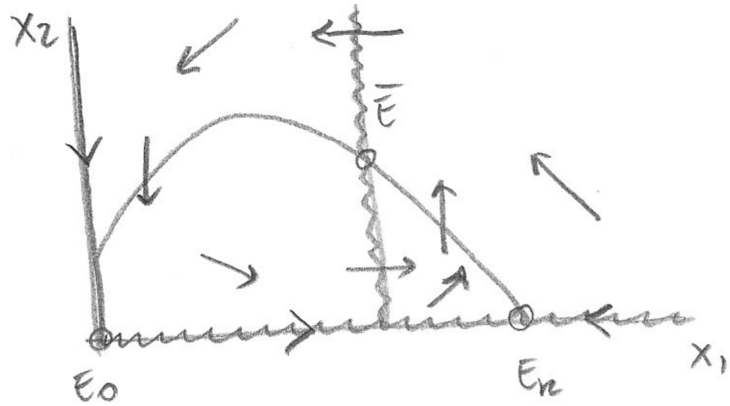
Traiettorie

$k < \frac{mb}{ea-m}$



E_0 INST
E_k STAB

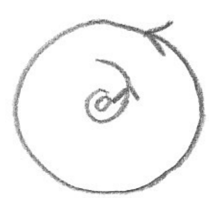
$k > \frac{mb}{ea-m}$



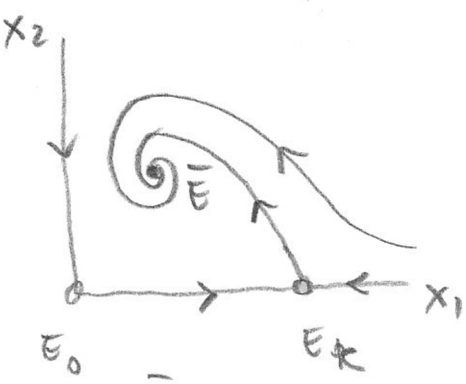
E_0, E_k INST

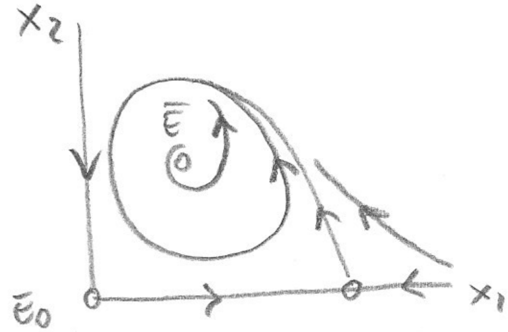
Nell'intorno di $\bar{E}$

oppure



$\Downarrow$ $\Downarrow$





$\bar{E}$ STAB

$\bar{E}$ INST

Stabilità

$$\dot{x}_1 = r x_1\left(1-\frac{x_1}{k}\right) - \frac{a x_1}{b+x_1} x_2 = x_1\left[r\left(1-\frac{x_1}{k}\right) - \frac{a}{b+x_1} x_2\right] = x_1 F_1(x_1, x_2) = f_1$$

$$\dot{x}_2 = e\frac{a x_1}{b+x_1} x_2 - m x_2 = x_2\left[\frac{e a x_1}{b+x_1} - m\right] = x_2 F_2(x_1) = f_2$$

All'equilibrio: $\begin{cases}\dot{x}_1 = 0\\ \dot{x}_2 = 0\end{cases} \rightarrow \begin{cases} x_1 F_1 = 0 \\ x_2 F_2 = 0\end{cases}$ da cui si hanno i tre equilib.

$\begin{cases} x_1 = 0 \\ x_2 = 0\end{cases} \rightarrow E_0 = (0,0)$ | $\begin{cases} x_2 = 0 \\ F_1 = 0\end{cases} \rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = k\end{array} \rightarrow E_k = (k, 0)$

$\begin{cases} F_1 = 0 \\ F_2 = 0\end{cases} \rightarrow \bar{E} = (\bar{x}_1, \bar{x}_2) \rightsquigarrow \in$ I quadrante se $0 < \frac{mb}{ea-m} < k$
(vedi isocline)

Stabilità via linearizzazione

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{vmatrix} = \begin{vmatrix} F_1 + x_1 \frac{\partial F_1}{\partial x_1} & x_1 \frac{\partial F_1}{\partial x_2} \\ x_2 \frac{\partial F_2}{\partial x_1} & F_2 + x_2 \underbrace{\cancel{\frac{\partial F_2}{\partial x_2}}}_{0}\end{vmatrix}$$

- Stabilità di $E_0 = (0,0)$

$$J_{E_0} = \begin{vmatrix} F_1(0,0) & 0 \\ 0 & F_2(0)\end{vmatrix} = \begin{vmatrix} r & 0 \\ 0 & -m\end{vmatrix}$$

$\lambda_1 = r > 0$ $\quad E_0$ INSTABILE
$\lambda_2 = -m < 0$ $\quad$ SELLA

- Stabilità di $E_k = (k, 0)$ $\quad \overset{\text{NOTA}}{\rightsquigarrow} F_1(k,0) = 0$ $\qquad \frac{\partial F_1}{\partial x_1} = -\frac{r}{k} + \frac{a x_2}{(b+x_1)^2}$

$$J_{E_k} = \begin{vmatrix} k\left(-\frac{r}{k}\right) & * \\ 0 & \frac{eak}{b+k} - m\end{vmatrix} = \begin{vmatrix} -r & * \\ 0 & \frac{eak}{b+k} - m\end{vmatrix}$$

$\lambda_1 = -r < 0$

$\lambda_2 = \frac{eak}{b+k} - m \rightsquigarrow$ il segno di λ_2 determina la stabilità di E_k

$\Rightarrow \lambda_2 < 0 \qquad \frac{eak}{b+k} - m < 0 \to eak - mb - mk < 0$

$\frac{mb}{ea-m} > k \qquad E_k$ è asint. stab.

$\Rightarrow \lambda_2 > 0 \quad \text{---} \quad \frac{mb}{ea-m} < k \qquad E_k$ è instabile $\rightsquigarrow$ sella

- Stabilità di $\bar{E} \overset{\text{NOTA}}{\rightsquigarrow} F_1(\bar{x}_1, \bar{x}_2) = 0$ e $F_2(\bar{x}_1) = 0$ con $\bar{x}_1 > 0$ e $\bar{x}_2 > 0$

$$J_{\bar{E}} = \begin{vmatrix} \bar{x}_1 \frac{\partial F_1}{\partial x_1} & \bar{x}_1 \frac{\partial F_1}{\partial x_2} \\ \bar{x}_2 \frac{\partial F_2}{\partial x_1} & 0 \end{vmatrix}$$

$$\frac{\partial F_1}{\partial x_1} = -\frac{r}{k} + \frac{a x_2}{(b+x_1)^2} \quad \Bigg| \quad \frac{\partial F_1}{\partial x_2} = -\frac{a}{b+x_1} \quad \Bigg| \quad \frac{\partial F_2}{\partial x_1} = \frac{eab}{(b+x_1)^2}$$

$$\text{tr}\, J_{\bar{E}} = \left[-\frac{r}{k} + \frac{a\bar{x}_2}{(b+\bar{x}_1)^2} \right] \bar{x}_1$$

$$\det J_{\bar{E}} = \bar{x}_1 \bar{x}_2 \left(+ \frac{a}{b+\bar{x}_1} \right) \left(\frac{eab}{(b+\bar{x}_1)^2} \right) > 0$$

Il segno di tr $J_{\bar{E}}$ determina la stabilità di $J_{\bar{E}}$

All'equilibrio $\bar{E} \in$ parabola $\to \bar{x}_2 = \frac{r}{a}\left(1 - \bar{x}_1/k\right)(b + \bar{x}_1)$

$$\Rightarrow \text{tr}\, J_{\bar{E}} = \left[-\frac{r}{k} + \frac{r\left(1 - \frac{\bar{x}_1}{k}\right)\cancel{(b+\bar{x}_1)}}{(b+\bar{x}_1)^{\cancel{2}}} \right] \bar{x}_1$$

$$\text{segno}\left(\text{tr}\, J_{\bar{E}}\right) \propto -1 + \frac{k - \bar{x}_1}{b + \bar{x}_1}$$

$\mathrm{tr}\, J_{\bar{E}} < 0 \qquad -1 + \dfrac{k - \bar{x}_1}{b + \bar{x}_1} < 0$

$$-b - \bar{x}_1 + k - \bar{x}_1 < 0$$

$$\bar{x}_1 > \frac{k-b}{2} \Rightarrow \begin{cases} \underbrace{\dfrac{mb}{ea-m}}_{\bar{x}_1} > \dfrac{k-b}{2} \\ \dfrac{mb}{ea-m} < k \end{cases}$$

$\bar{x}_1 > 0$

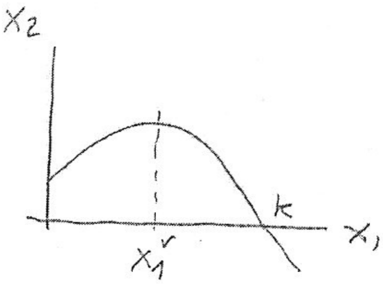
$$\Rightarrow \frac{k-b}{2} < \frac{mb}{ea-m} < k \qquad \bar{E} \text{ è as. stab.}$$

$\mathrm{tr}\, J_{\bar{E}} > 0 \quad \dots \quad \dfrac{mb}{ea-m} < \dfrac{k-b}{2} \qquad \bar{E}$ è instabile

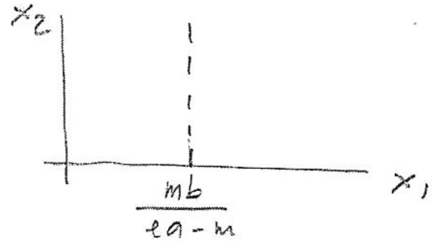
Riassumendo:

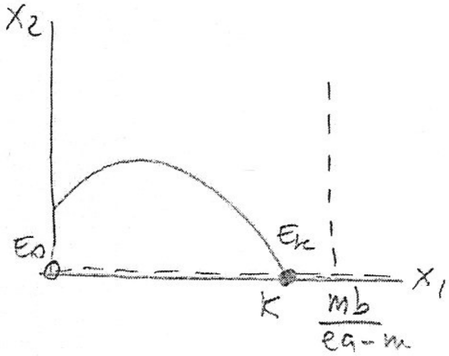
$\dot{x}_1 = 0 \xrightarrow[\text{banale}]{\text{non}} x_2 = \frac{r}{a}(b + x_1)\left(1 - \frac{x_1}{k}\right)$

$x_1^v = \frac{k-b}{2}$



$\dot{x}_2 = 0 \xrightarrow[\text{banale}]{\text{non}} x_1 = \frac{mb}{ea - m}$

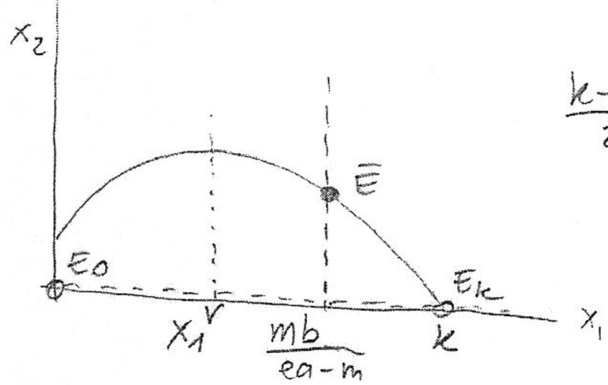


- 

$\frac{mb}{ea-m} > k$

$E_k = (k, 0)$ stab

$E_0 = (0, 0)$ sella

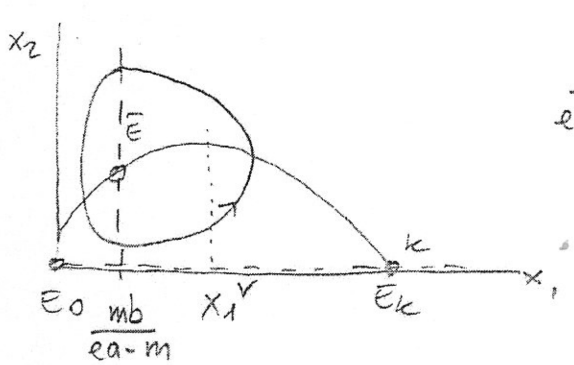
- 

$\frac{k-b}{2} < \frac{mb}{ea-m} < k$

$E_k = (k, 0)$ sella

$E_0 = (0, 0)$ sella

$\bar{E} = (\bar{x}_1, \bar{x}_2)$ stabile

- 

$\frac{mb}{ea-m} < \frac{k-b}{2}$

$E_k = (k, 0)$ sella

$E_0 = (0, 0)$ sella

$\bar{E} = (\bar{x}_1, \bar{x}_2)$ instabile

$\exists$ ciclo stabile

Modello di competizione interspecifica

x_1 = densità di batteri utili

x_2 = densità di batteri nocivi

$$\dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{k_1}\right) - \alpha_{12} x_1 x_2 \longrightarrow$$

$$\dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{k_2}\right) - \alpha_{21} x_1 x_2 \longrightarrow$$

extra-mortalità per competizione interspecifica

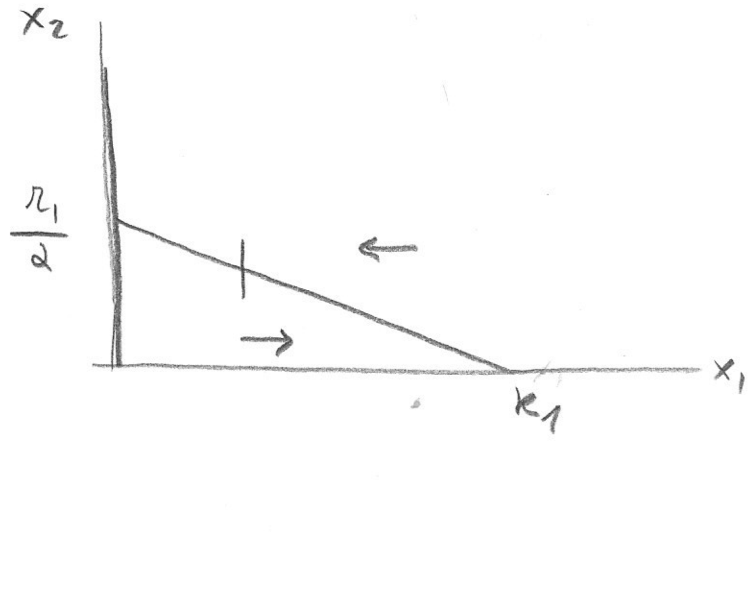
Supponiamo (per semplicità di calcolo) $\alpha_{12} = \alpha_{21} = \alpha$

$$\dot{x}_1 = x_1 \left[r_1 \left(1 - \frac{x_1}{k_1}\right) - \alpha x_2 \right] = x_1 F_1(x_1, x_2)$$

$$\dot{x}_2 = x_2 \left[r_2 \left(1 - \frac{x_2}{k_2}\right) - \alpha x_1 \right] = x_2 F_2(x_1, x_2)$$

<u>Isocline</u> $\dot{x}_1 = 0$

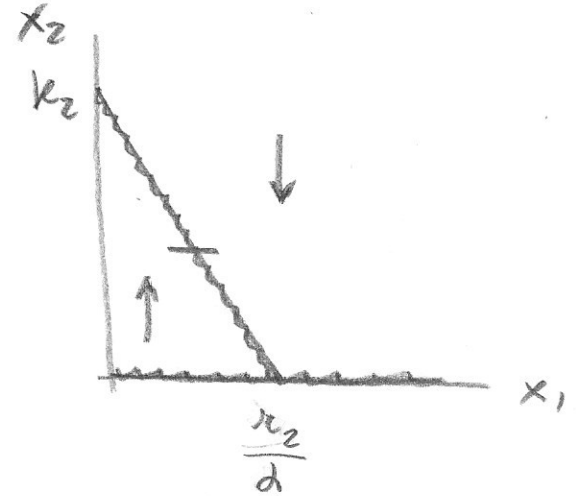
- $x_1 = 0$
- $x_2 = \frac{r_1}{\alpha}\left(1 - \frac{x_1}{k_1}\right) \rightsquigarrow F_1 = 0$



Isocline $\dot{x}_2 = 0$

$$x_2 = 0$$

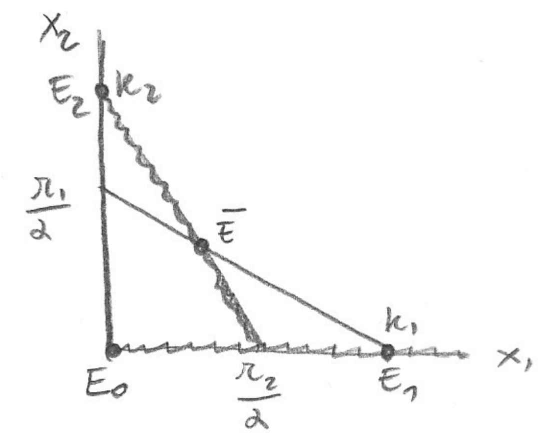
$$x_1 = \frac{r_2}{\alpha}\left(1 - \frac{x_2}{k_2}\right) \rightsquigarrow F_2 = 0$$



Equilibri $\cap$ isocline $\dot{x}_1 = 0$ e $\dot{x}_2 = 0$

- $\begin{cases} k_1 > \frac{r_2}{\alpha} \\ k_2 > \frac{r_1}{\alpha} \end{cases}$

(α grande)



$E_0 = (0,0)$

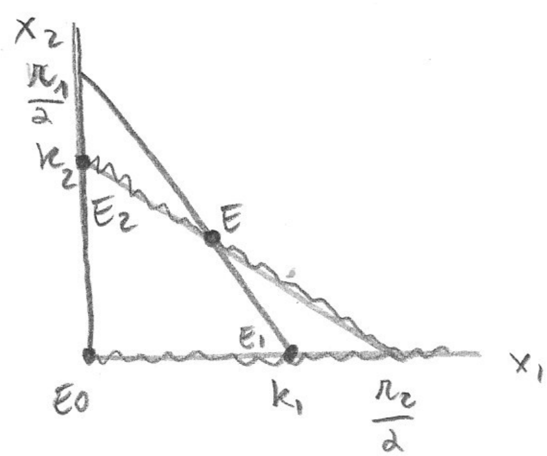
$E_1 = (k_1, 0)$

$E_2 = (0, k_2)$

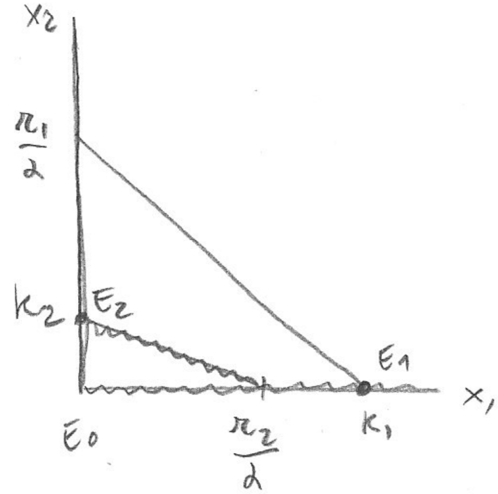
$\bar{E} = (\bar{x}_1, \bar{x}_2)$ t.c. $\begin{cases} F_1 = 0 \\ F_2 = 0 \end{cases}$

- $\begin{cases} k_1 < \frac{r_2}{\alpha} \\ k_2 < \frac{r_1}{\alpha} \end{cases}$

(α piccolo)

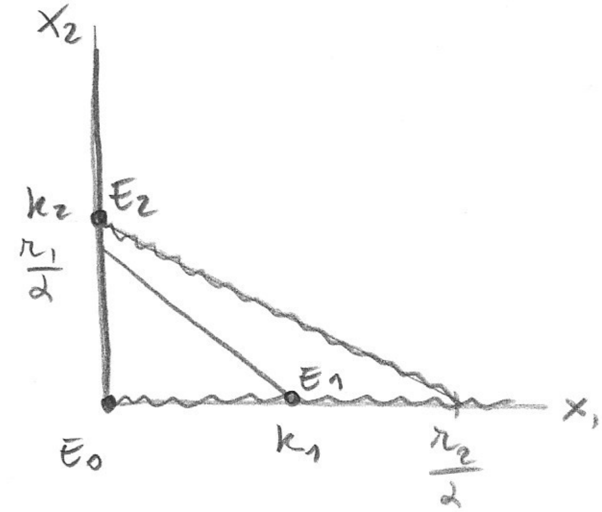


- $\begin{cases} k_1 > \frac{r_2}{\alpha} \\ k_2 < \frac{r_1}{\alpha} \end{cases}$



$\nexists \bar{E}$

- $\begin{cases} k_1 < \frac{r_2}{\alpha} \\ k_2 > \frac{r_1}{\alpha} \end{cases}$

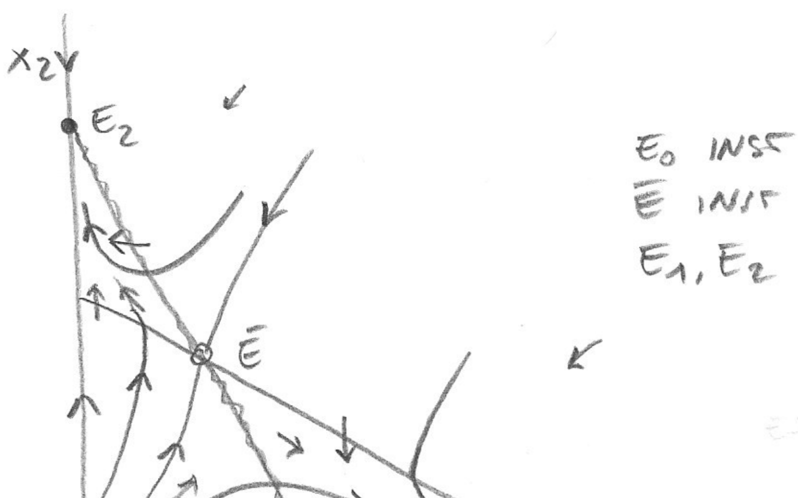


$\nexists \bar{E}$

<u>Traiettorie</u>

<u>NOTA</u>: Gli assi sono invarianti

- $\begin{cases} k_1 > \frac{r_2}{\alpha} \\ k_2 > \frac{r_1}{\alpha} \end{cases}$

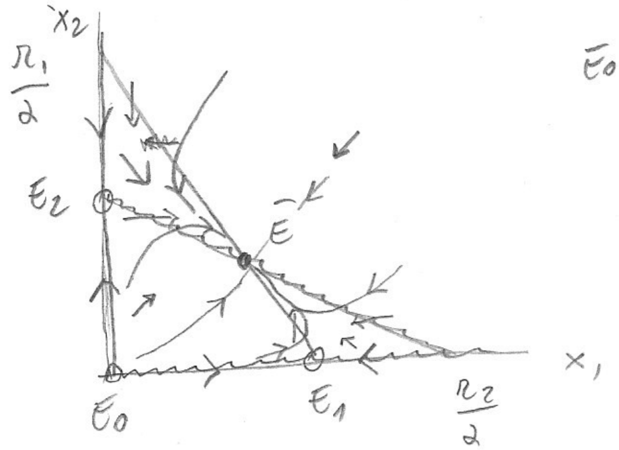


E_0 INST (REP)
$\bar{E}$ INST (SELLA)
E_1, E_2 STAB

Esclusione competitiva

$\left(\begin{array}{l} r_1 = r_2 = s \\ \ldots \end{array} \right) \rightarrow \frac{r_1}{\alpha} = \frac{r_2}{\alpha} = \frac{1}{2}$

• $\begin{cases} k_1 < \frac{r_2}{2} \\ k_2 < \frac{r_1}{2} \end{cases}$

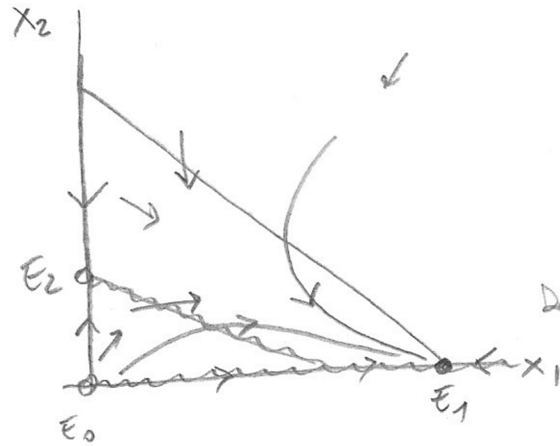


E_0, E_1, E_2 INST

$\bar{E}$ STAB

Coesistenza competitiva

• $\begin{cases} k_1 > \frac{r_2}{2} \\ k_2 < \frac{r_1}{2} \end{cases}$



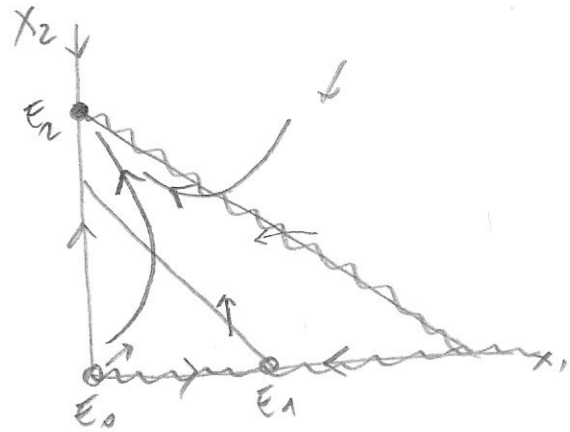
E_0, E_2 INST

E_1 STAB

Vince la competizione la specie con isoclina "più alta"

DOMINANZA 1

• $\begin{cases} k_1 < \frac{r_2}{2} \\ k_2 > \frac{r_1}{2} \end{cases}$

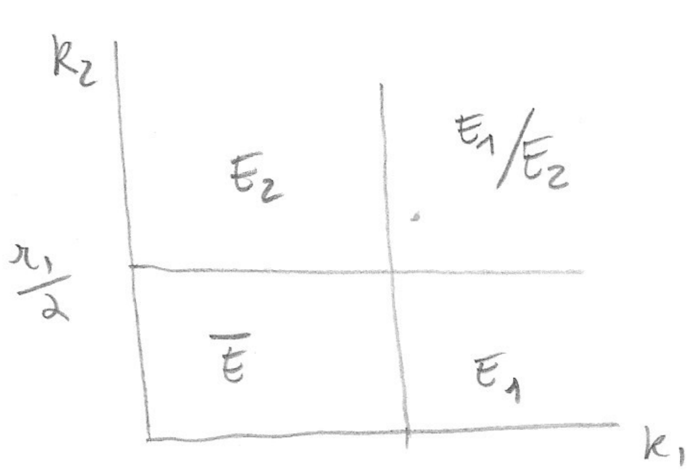


E_0, E_1 INST

E_2 STAB

DOMINANZA 2

EQ. STABILI



Studio della stabilità via linearizzazione

$$\dot{x}_1 = x_1 F_1(x_1, x_2) = f_1(x_1, x_2)$$
$$\dot{x}_2 = x_2 F_2(x_1, x_2) = f_2(x_1, x_2)$$

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} F_1 + x_1 \frac{\partial F_1}{\partial x_1} & x_1 \frac{\partial F_1}{\partial x_2} \\ x_2 \frac{\partial F_2}{\partial x_1} & F_2 + x_2 \frac{\partial F_2}{\partial x_2} \end{vmatrix}$$

$$\frac{\partial F_1}{\partial x_1} = -\frac{r_1}{k_1} \qquad \frac{\partial F_1}{\partial x_2} = -\alpha$$

$$\frac{\partial F_2}{\partial x_1} = -\alpha \qquad \frac{\partial F_2}{\partial x_2} = -\frac{r_2}{k_2}$$

$$E_0 = (0,0) \rightarrow J\Big|_{E_0} = \begin{vmatrix} F_1(0,0) & 0 \\ 0 & F_2(0,0) \end{vmatrix} = \begin{vmatrix} r_1 & 0 \\ 0 & r_2 \end{vmatrix}$$

$\lambda_1 = r_1 \quad \lambda_2 = r_2 \quad \lambda_i > 0 \rightarrow$ INST (REPULS)

$$E_1 = (k_1, 0) \rightarrow J\Big|_{E_1} = \begin{vmatrix} -r_1 & * \\ 0 & r_2 - \alpha k_1 \end{vmatrix} \qquad \begin{aligned} \lambda_1 &= -r_1 < 0 \\ \lambda_2 &= r_2 - \alpha k_1 \end{aligned}$$

$\Downarrow$

$F_1(E_1) = 0$

$r_2 - \alpha k_1 < 0 \rightarrow k_1 > \frac{r_2}{\alpha} \rightarrow \lambda_2 < 0 \Rightarrow E_1$ STAB

$r_2 - \alpha k_1 > 0 \rightarrow k_1 < \frac{r_2}{\alpha} \rightarrow \lambda_2 > 0 \Rightarrow E_1$ INST

$E_2 = (0, k_2) \rightsquigarrow$ come sopra $\rightsquigarrow k_2 > \frac{r_1}{\alpha} \Rightarrow E_2$ STAB

$k_2 < \frac{r_1}{\alpha} \Rightarrow E_2$ INST

$$\bar{E} = (\bar{x}_1, \bar{x}_2) \rightarrow J\Big|_{\bar{E}} = \begin{vmatrix} \bar{x}_1\left(-\frac{r_1}{k_1}\right) & \bar{x}_1(-\alpha) \\ \bar{x}_2(-\alpha) & \bar{x}_2\left(-\frac{r_2}{k_2}\right) \end{vmatrix} \quad \text{con } \bar{x}_1, \bar{x}_2 > 0$$

$\downarrow$

$F_1 = F_2 = 0$

$tr < 0$

$\det = \bar{x}_1 \bar{x}_2 \left(\frac{r_1 r_2}{k_1 k_2} - \alpha^2 \right)$

$\det > 0 \Rightarrow \frac{r_1 r_2}{k_1 k_2} - \alpha^2 > 0 \Rightarrow \bar{E}$ è STAB

$\exists \bar{E}$ solo per $\begin{cases} k_1 < \frac{r_2}{\alpha} \\ k_2 < \frac{r_1}{\alpha} \end{cases} \rightsquigarrow \frac{r_1 r_2}{\alpha^2} > k_1 k_2 \rightsquigarrow \frac{r_1 r_2}{k_1 k_2} > \alpha^2 \rightsquigarrow \det > 0$

$\Downarrow$

$\bar{E}$ STAB

oppure per $\begin{cases} k_1 > \frac{r_2}{\alpha} \\ k_2 > \frac{r_1}{\alpha} \end{cases} \rightsquigarrow \cdots \rightsquigarrow \det < 0 \Rightarrow \bar{E}$ INST