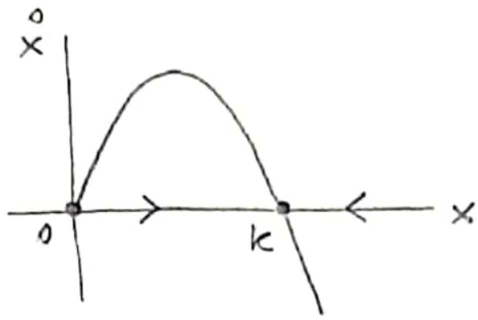
Crescita logistica di una singola risorsa (x)

$x(t)$ = densità / biomassa della risorsa

$$\dot{x} = rx\left(1-\frac{x}{k}\right) \qquad r, k > 0$$

r = tasso intrinseco di crescita

k = capacità portante



Equilibri $\dot{x} = 0$ — $\bar{x} = 0$, $\bar{\bar{x}} = k$

Risorsa (x) – Consumatore costante (P) (preda – predatore costante)

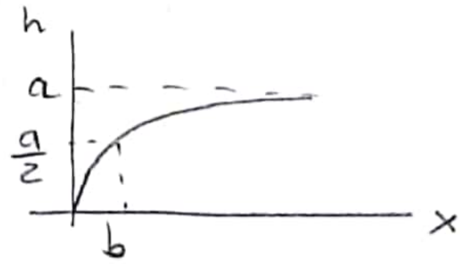
$$\textcircled{x} \longrightarrow \boxed{P}$$

$$\dot{x} = \underbrace{rx\left(1-\frac{x}{k}\right)}_{\text{crescita logistica}} - \underbrace{\frac{ax}{b+x}P}_{\text{mortalità per predazione}}$$

$$h(x) = \frac{ax}{b+x}$$

Risposta funzionale di Holling di tipo II

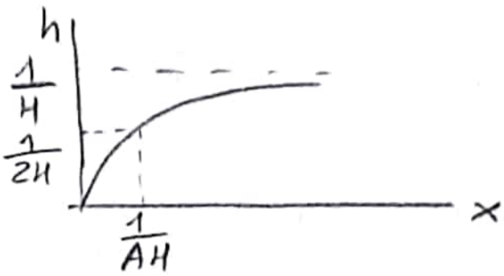
$$[h] = \frac{[x]}{[P][t]}$$



a = massima capacità predatoria

b = costante di semi-saturazione

In alternativa $h(x) = \dfrac{Ax}{1+AHx}$



A = attack rate

H = handling time

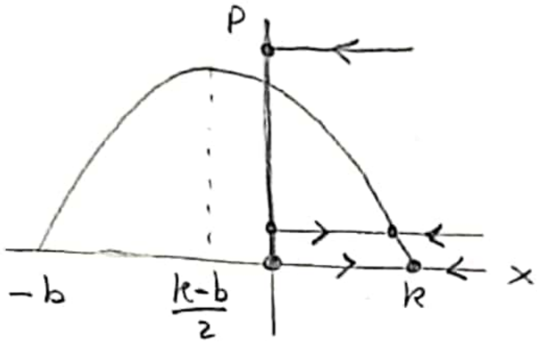
$$a = \frac{1}{H} \qquad b = \frac{1}{AH} \longrightarrow A = \frac{a}{b} \qquad H = \frac{1}{a}$$

Equilibri $\dot{x}=0$ ⟨ $x=0$

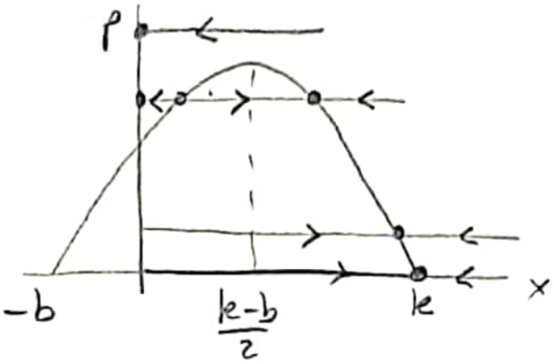
$r\left(1-\frac{x}{k}\right)-\frac{a}{b+x}p=0$

$$p=\frac{r}{a}\left(1-\frac{x}{k}\right)(b+x)$$

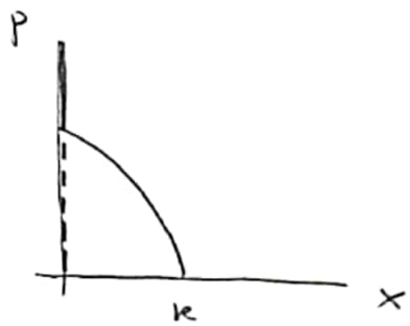
$b>k$



$b<k$



⇓



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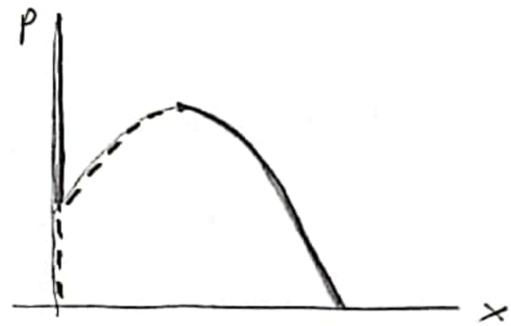


Risorsa – Consumatore
(x) (y)

Preda – Predatore

$x \to y$

Modello di Rosenzweig-MacArthur

$$\dot{x}=rx\left(1-\frac{x}{k}\right)-a\frac{x}{b+x}y$$

crescita logistica   mortalità per predazione

$$\dot{y}=e\frac{ax}{b+x}y-my$$

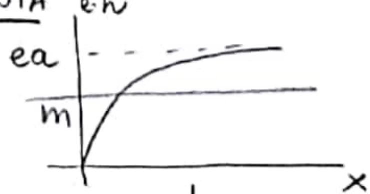
crescita

efficienza di conversione

mortalità naturale

tasso di mortalità

NOTA



$ea>m$ cosicché y può crescere se $x\to\infty$

Equilibri $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$

$$\dot{x} = 0 \quad \begin{cases} x = 0 \\ y = \frac{r}{a}\left(1 - \frac{x}{k}\right)(b+x) \end{cases}$$

$$\dot{y} = 0 \quad \begin{cases} y = 0 \\ x = \frac{mb}{ea - m} \end{cases}$$

$E_0 = (0, 0) \qquad E_k = (k, 0) \qquad E = (\bar{x}, \bar{y})$

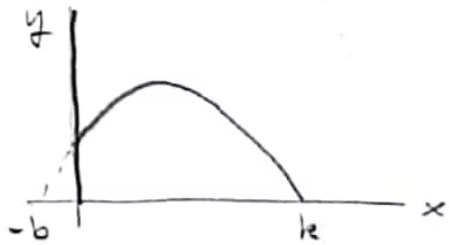
$\bar{x} = \frac{mb}{ea-m}$, $\quad \bar{y} = \frac{r}{a}\left(1 - \frac{\bar{x}}{k}\right)(b + \bar{x})$

con $1 - \frac{\bar{x}}{k} > 0$

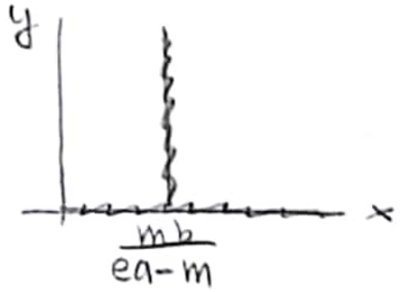
$\Rightarrow \bar{x} < k \rightarrow \frac{mb}{ea-m} < k$

Equilibri $\rightarrow \cap$ isocline

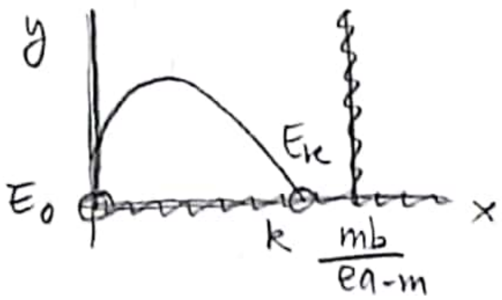
$$\underline{\dot{x} = 0} \quad \begin{cases} x = 0 \\ y = \frac{r}{a}\left(1 - \frac{x}{k}\right)(b+x) \end{cases}$$



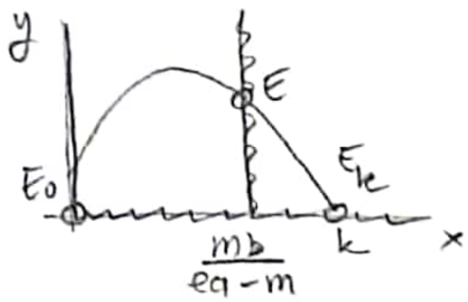
$$\dot{y} = 0 \quad \begin{cases} y = 0 \\ x = \frac{mb}{ea-m} \end{cases}$$



$\frac{mb}{ea-m} > k$



$\frac{mb}{ea-m} < k$



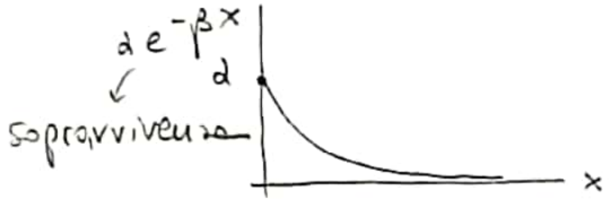
Modello di Ricker → crescita di una singola risorsa a tempo discreto $x(t)$

↓

Salmoni dell'Oceano Pacifico
(popolazione semelpara)

> NOTA: $\alpha > 1$
> x piccolo $e^{-\beta x} \sim 1$
> $x(t+1) = \alpha x(t)$ e $x(t)$ cresce
> se $\alpha < 1$ $x(t)$ decresce

$$x(t+1) = \alpha x(t) e^{-\beta x(t)} = f(x(t))$$



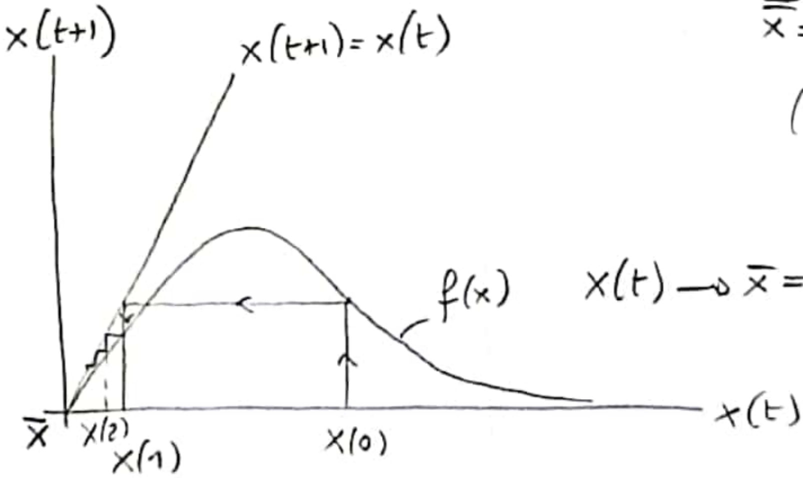
Equilibri: $x(t+1) = x(t) = \bar{x}$

$$\bar{x} = \alpha \bar{x} e^{-\beta \bar{x}} \quad \begin{cases} \bar{x} = 0 \\ 1 = \alpha e^{-\beta \bar{x}} \to \bar{\bar{x}} = \frac{1}{\beta} \ln \alpha \quad (\text{con } \alpha > 1) \end{cases}$$

Diagramma di Moran

$\alpha < 1$ $(\nexists \bar{\bar{x}})$

$(f'(0) = \alpha < 1)$



$x(t) \to \bar{x} = 0$

$\alpha > 1$? Dipende! $1 < \alpha < e^2 \Rightarrow x(t) \to \bar{\bar{x}}$
$\alpha > e^2 \Rightarrow x(t) \to$ qualcosa di più complicato
- ciclo
- caos

vedi simulatore

```matlab
function [stato]=ricker(alfa,beta,x0)
% simulazione modello di Ricker
% [stato]=ricker(alfa,beta,x0)
% [stato]=ricker(0.8,1,0.2); % ---> equilibrio nullo
% [stato]=ricker(6,1,0.2); % ---> equilibrio positivo
% [stato]=ricker(10,1,0.2); % ---> ciclo di periodo 2
% [stato]=ricker(13,1,0.2); % ---> ciclo di periodo 4
% [stato]=ricker(14.5,1,0.2); % ---> ciclo di periodo 8
% [stato]=ricker(15,1,0.2); % ---> caos
% [stato]=ricker(20,1,0.2);  % ---> caos
% [stato]=ricker(23,1,0.2); % ---> ciclo di periodo 3
% [stato]=ricker(24,1,0.2); % ---> ciclo di periodo 6

close all;

%%%%%%%%%%%% SIMULAZIONE %%%%%%%%%%%%
tfin=30; stato=x0;
for ind=2:tfin
    x=stato(ind-1);
    stato=[stato;alfa*x*exp(-beta*x)];
end;
%%%%%%%%%%%% MOVIMENTO %%%%%%%%%%%%
figure;
plot(stato,'-*');
xlabel( 'tempo' )
ylabel( 'x(t)' )
pause;
%%%%%%%%%%%% COSTRUZIONE DEL DIAGRAMMA DI MORAN "DINAMICO" %%%%%%%%%%%%
moran1=ones(2*tfin,1);
moran2=ones(2*tfin,1);
for h=1:tfin
    moran1(2*h-1)=stato(h);
    moran1(2*h)=stato(h);
end;
moran2=[0;moran1(3:2*tfin);1];
figure;
hold on
massimo=1.2*max(x0,alfa/beta*exp(-1));
x=0:massimo/100:massimo;
y=alfa*x.*exp(-beta*x);
plot(x,y,'r');
plot(x,x,'k');
xlabel( 'x(t)' )
ylabel( 'x(t+1)' )
title('diagramma di Moran con transitorio');
for ind=1:length(moran1)-2
    plot([moran1(ind) moran1(ind+1)],[moran2(ind) moran2(ind+1)],'b-');
    pause(0.2)
end;
pause;
% Diagramma di Moran a transitorio esaurito
tfin=200;
stato=x0;
for ind=2:tfin
    x=stato(ind-1);
    stato=[stato;alfa*x*exp(-beta*x)];
end;
stato=stato(length(stato));
for ind=2:tfin
    x=stato(ind-1);
    stato=[stato;alfa*x*exp(-beta*x)];
end;
figure;
plot(stato(1:length(stato)-1),stato(2:length(stato)),'*');
hold on;
massimo=1.2*max(x0,alfa/beta*exp(-1));
x=0:massimo/100:massimo;
y=alfa*x.*exp(-beta*x);
plot(x,y,'r');
xlabel( 'x(t)' )
ylabel( 'x(t+1)' )
title('diagramma di Moran su attrattore (transitorio esaurito)');
moran1=ones(2*tfin,1);
moran2=ones(2*tfin,1);
for h=1:tfin
    moran1(2*h-1)=stato(h);
    moran1(2*h)=stato(h);
end;
moran2=[0;moran1(3:2*tfin);1];
plot(moran1(1:length(moran1)-1),moran2(1:length(moran1)-1),'-.');
plot(x,x,'k');
pause;
% ----------------
figure;
hold on;
axis([0 massimo -1 1]);
plot([0 massimo],[0 0],'k');
axis('off');
text(massimo,-0.1,'x(t)');
text(0,-0.1,'0');
title('attrattore');
plot(stato(1),0,'ro','markersize',5);
pause;
for ind=1:length(stato)
    plot(stato(ind),0,'b*','markersize',5);
    pause(0.1)
end;
```

Modelli di epidemie

SI

susc**e**ttibili — infetti

$$S \xrightarrow{\alpha} I \xrightarrow{\beta}$$

Immunizzazione permanente
(morbillo/rosolia/varicella ...)

$$\dot{S} = -\overbrace{\alpha S I}^{\text{contagio}}$$
$$\dot{I} = \alpha S I - \underbrace{\beta I}_{\text{guarigione}}$$

$\alpha \downarrow$ con mascherine/vaccini

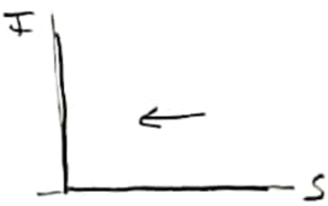
$\beta \uparrow$ con cure

Equilibri

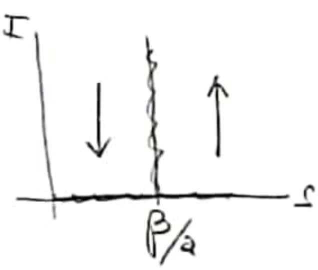
$$\dot{S} = 0 \begin{cases} S = 0 \\ I = 0 \end{cases} \qquad \dot{I} = 0 \begin{cases} I = 0 \\ S = \beta/\alpha \end{cases} \quad \Rightarrow \exists \infty \text{ equilibri } (\bar{S}, 0)$$

Isocline

$\dot{S} = 0$

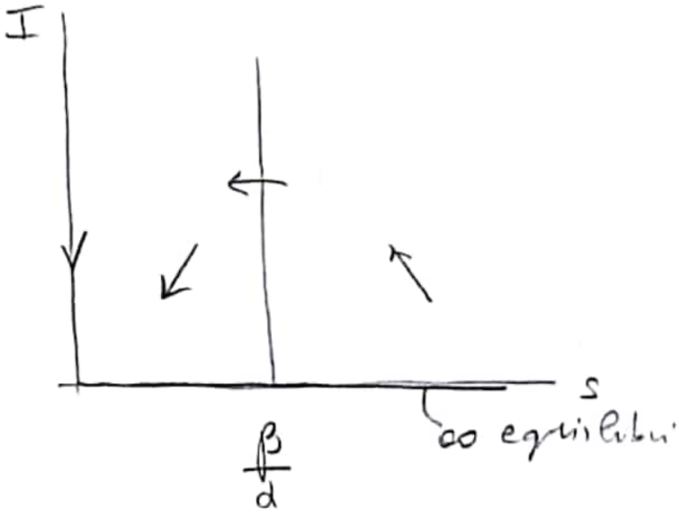


$\dot{I} = 0$



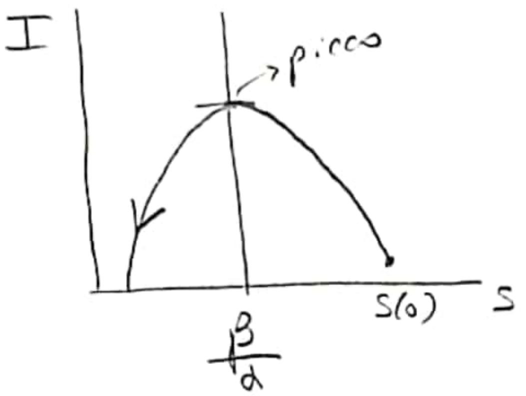
Equilibri ⇒ ∩ isocline ⇒ $I = 0 \ \forall S$

Vettore tangente



$S(0) > \frac{\beta}{\alpha}$ → L'epidemia "parte"

$I(0) > 0$

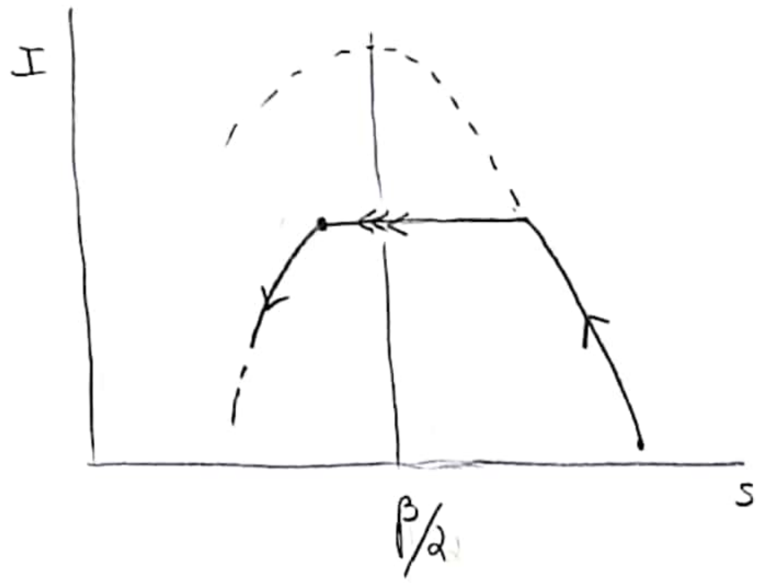


$S(0) < \frac{\beta}{\alpha}$ → L'epidemia "non parte"

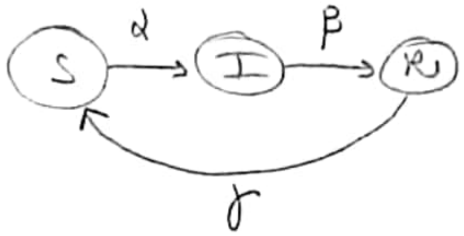
Per fare in modo che $S(0)$ sia $< \frac{\beta}{\alpha}$ → β alto (cure)
α piccolo (mascherine, vaccini)

Lockdown?

$S \downarrow$ sotto β/α



SIR — suscettibili, infetti, rimossi



Immunizzazione temporanea (influenza/pertosse...)

$$\dot{S} = -\alpha S I + \gamma R$$
$$\dot{I} = \alpha S I - \beta I$$
$$\dot{R} = \beta I - \gamma R$$

$\nexists$ morti (opp. trascurabili)

$S + I + R = N \quad \forall t$

N → popolazione totale

Equilibrio

$$\begin{cases} -\alpha S I + \gamma R = 0 \\ \alpha S I - \beta I = 0 \\ \beta I - \gamma R = 0 \end{cases} \longrightarrow I = 0 \to R = 0 \to \forall S \qquad \Rightarrow \bar{E} = (\bar{S}, 0, 0)$$

oppure $S = \frac{\beta}{\alpha} \qquad R = \frac{\beta}{\gamma} I$

$$\frac{\beta}{\alpha} + I + \frac{\beta}{\gamma} I = N \longrightarrow I = \frac{N - \frac{\beta}{\alpha}}{1 + \beta/\gamma} = \frac{\gamma}{\alpha} \frac{\alpha N - \beta}{\gamma + \beta}$$

$$\Rightarrow \bar{\bar{E}} = \left(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha} \frac{\alpha N - \beta}{\gamma + \beta}, \frac{\beta}{\alpha} \frac{\alpha N - \beta}{\gamma + \beta} \right)$$

SIRV