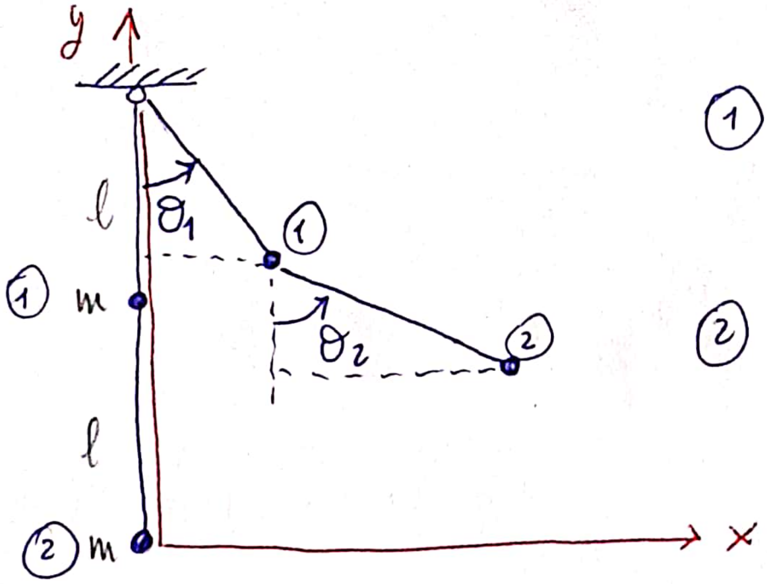
9. Dinamica del pendolo doppio



① $x_1 = l \,\text{sen}\,\theta_1$ posizione

$y_1 = l(-\cos\theta_1 + 2)$ $(\dot{x}_1, \dot{y}_1)$ velocità v_1

② $x_2 = l(\text{sen}\,\theta_1 + \text{sen}\,\theta_2)$ posizione

$y_2 = l(2 - \cos\theta_1 - \cos\theta_2)$

$(\dot{x}_2, \dot{y}_2)$ velocità v_2

$$\dot{x}_1 = l \cos\theta_1 \dot{\theta}_1$$

$$\dot{y}_1 = l \,\text{sen}\,\theta_1 \dot{\theta}_1$$

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l^2 \cos^2\theta_1 \dot{\theta}_1^2 + l^2 \text{sen}^2\theta_1 \dot{\theta}_1^2 =$$

$$= l^2 \underbrace{(\cos^2\theta_1 + \text{sen}^2\theta_1)}_{1} \dot{\theta}_1^2 = l^2 \dot{\theta}_1^2$$

$$\dot{x}_2 = l(\cos\theta_1 \dot{\theta}_1 + \cos\theta_2 \dot{\theta}_2)$$

$$\dot{y}_2 = l(\text{sen}\,\theta_1 \dot{\theta}_1 + \text{sen}\,\theta_2 \dot{\theta}_2)$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l^2(\cos^2\theta_1 \dot{\theta}_1^2 + \cos^2\theta_2 \dot{\theta}_2^2 + 2\cos\theta_1\cos\theta_2 \dot{\theta}_1 \dot{\theta}_2) +$$

$$+ l^2(\text{sen}^2\theta_1 \dot{\theta}_1^2 + \text{sen}^2\theta_2 \dot{\theta}_2^2 + 2\,\text{sen}\,\theta_1 \text{sen}\,\theta_2 \dot{\theta}_1 \dot{\theta}_2) =$$

$$= l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2 \underbrace{(\cos\theta_1\cos\theta_2 + \text{sen}\,\theta_1 \text{sen}\,\theta_2)}_{\cos(\theta_1 - \theta_2)} \dot{\theta}_1\dot{\theta}_2 =$$

$$= l^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right]$$

Energia cinetica totale $T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$

$$T = \frac{1}{2} m l^2 \dot{\vartheta}_1^2 + \frac{1}{2} m l^2 \left[\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \right] =$$

$$= \frac{1}{2} m l^2 \left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \right] =$$

$$= m l^2 \left[\dot{\vartheta}_1^2 + \frac{1}{2} \dot{\vartheta}_2^2 + \cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \right]$$

tra t_1 e t_2, $\vartheta_1, \dot{\vartheta}_1, \vartheta_2, \dot{\vartheta}_2$ variano $\Rightarrow$ varia anche T

$$\left.\begin{matrix} \delta\vartheta_1 & \delta\dot{\vartheta}_1 \\ \delta\vartheta_2 & \delta\dot{\vartheta}_2 \end{matrix}\right| \Rightarrow \delta T$$

differenziale ◻

$$\delta T = m l^2 \Big[2\dot{\vartheta}_1 \delta\dot{\vartheta}_1 + \frac{1}{\cancel{2}} \cancel{2} \dot{\vartheta}_2 \delta\dot{\vartheta}_2 - \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_1 +$$
$$+ \cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_2 \delta\dot{\vartheta}_1 + \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_2 +$$
$$+ \cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \delta\dot{\vartheta}_2 \Big] =$$

$$= m l^2 \Big[-\operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_1 + \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_2 +$$
$$+ \left(2\dot{\vartheta}_1 + \cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_2\right) \delta\dot{\vartheta}_1 + \left(\dot{\vartheta}_2 + \cos(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1\right) \delta\dot{\vartheta}_2 \Big]$$

(*)

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} m l^2 \left[\cancel{-\operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_1} + \cancel{\operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2 \delta\vartheta_2} \right] dt +$$

$$- \int_{t_1}^{t_2} m l^2 \left[2\ddot{\vartheta}_1 \cancel{- \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2} + \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_2^2 + \cos(\vartheta_1 - \vartheta_2) \ddot{\vartheta}_2 \right] \delta\vartheta_1 \, dt +$$

$$- \int_{t_1}^{t_2} m l^2 \left[\ddot{\vartheta}_2 - \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1^2 \cancel{+ \operatorname{sen}(\vartheta_1 - \vartheta_2) \dot{\vartheta}_1 \dot{\vartheta}_2} + \cos(\vartheta_1 - \vartheta_2) \ddot{\vartheta}_1 \right] \delta\vartheta_2 \, dt$$

△ integrazione per parti

□

δT = variazione dell'energia cinetica

$$T = T(\vartheta_1, \vartheta_2, \dot\vartheta_1, \dot\vartheta_2)$$

$$\delta T = \frac{\partial T}{\partial \vartheta_1}\delta\vartheta_1 + \frac{\partial T}{\partial \vartheta_2}\delta\vartheta_2 + \frac{\partial T}{\partial \dot\vartheta_1}\delta\dot\vartheta_1 + \frac{\partial T}{\partial \dot\vartheta_2}\delta\dot\vartheta_2$$

⊛

Le variazioni $\delta\dot\vartheta_1$ e $\delta\dot\vartheta_2$ sono nulle in corrispondenza di t_1 e t_2

$\Rightarrow$ t_1 e t_2 sono istanti di tempo in cui le velocità delle due masse sono nulle

Δ Integrazione per parti $\int f\,dg = f\cdot g - \int g\,df$

$$\int_a^b \underset{\downarrow}{f(x)}\,\underbrace{g'(x)\,dx} = f(b)\,g(b) - f(a)\,g(a) - \int f'(x)\,g(x)\,dx$$

$$(2\dot\vartheta_1 + \cos(\vartheta_1-\vartheta_2)\dot\vartheta_2)\cdot \delta\dot\vartheta_1\,dt$$

$$\Rightarrow \int_{t_1}^{t_2}(2\dot\vartheta_1 + \cos(\vartheta_1-\vartheta_2)\dot\vartheta_2)\,\delta\dot\vartheta_1\,dt = \cancel{(\dots)_{t=t_2}\,\dot\vartheta_1\Big|_{t=t_2} - (\dots)_{t=t_1}\cdot\dot\vartheta_1\Big|_{t=t_1}}$$

$$- \int_{t_1}^{t_2}\left[2\ddot\vartheta_1 - \operatorname{sen}(\vartheta_1-\vartheta_2)\dot\vartheta_1\dot\vartheta_2 + \operatorname{sen}(\vartheta_1-\vartheta_2)\dot\vartheta_2^2 + \cos(\vartheta_1-\vartheta_2)\ddot\vartheta_2\right]\delta\vartheta_1\,dt$$

ecc per il secondo integrale

$$\int_{t_1}^{t_2} \delta T\, dt = -ml^2 \int_{t_1}^{t_2} \left[2\ddot{\theta}_1 + \mathrm{sen}(\theta_1-\theta_2)\dot{\theta}_2^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_2\right]\delta\theta_1\, dt$$

$$-ml^2 \int_{t_1}^{t_2} \left[\ddot{\theta}_2 - \mathrm{sen}(\theta_1-\theta_2)\dot{\theta}_1^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_1\right]\delta\theta_2\, dt$$

Energia potenziale totale

$$U = mgy_1 + mgy_2 = mgl(2-\cos\theta_1) + mgl(2-\cos\theta_1-\cos\theta_2)$$

$$U = mgl(4-2\cos\theta_1-\cos\theta_2)$$

$$\delta U = 2mgl\,\mathrm{sen}\theta_1\,\delta\theta_1 + mgl\,\mathrm{sen}\theta_2\,\delta\theta_2$$

$$\int_{t_1}^{t_2} \delta U\, dt = mgl\int_{t_1}^{t_2}\left[2\,\mathrm{sen}\theta_1\,\delta\theta_1 + \mathrm{sen}\theta_2\,\delta\theta_2\right]dt$$

$$\int_{t_1}^{t_2} \delta(T-U)\,dt = 0 \Rightarrow$$

Principio di Hamilton

$\int_{t_1}^{t_2} \delta(T-U)\,dt = 0$ se tutte le forze sono conservative

$$\Rightarrow \cancel{ml^2}\int_{t_1}^{t_2}\left[2\ddot{\theta}_1 + \mathrm{sen}(\theta_1-\theta_2)\dot{\theta}_2^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_2\right]\delta\theta_1\,dt +$$

$$+\cancel{ml^2}\int_{t_1}^{t_2}\left[\ddot{\theta}_2 - \mathrm{sen}(\theta_1-\theta_2)\dot{\theta}_1^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_1\right]\delta\theta_2\,dt +$$

$$+\cancel{mgl}\,\frac{g}{l}\int_{t_1}^{t_2}\left[2\,\mathrm{sen}\theta_1\,\delta\theta_1 + \mathrm{sen}\theta_2\,\delta\theta_2\right]dt = 0$$

$$g/l = \omega^2$$

$$\int_{t_1}^{t_2}\left[2\ddot{\theta}_1 + \text{sen}(\theta_1-\theta_2)\dot{\theta}_2^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_2 + 2\omega^2\,\text{sen}\,\theta_1\right]\delta\theta_1\,dt\ +$$

$$+\int_{t_1}^{t_2}\left[\ddot{\theta}_2 - \text{sen}(\theta_1-\theta_2)\dot{\theta}_1^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_1 + \omega^2\,\text{sen}\,\theta_2\right]\delta\theta_2\,dt = 0$$

$\forall\delta\theta_1, \forall\delta\theta_2 \rightarrow$ $\delta\theta_2 = 0 \Rightarrow 1^\circ$ integrale $= 0 \overset{\forall\delta\theta_1}{\Rightarrow} [\dots] = 0$

$\delta\theta_1 = 0 \Rightarrow 2^\circ$ " $= 0 \overset{\forall\delta\theta_2}{\Rightarrow} [\dots] = 0$

$$\Rightarrow \begin{cases} 2\ddot{\theta}_1 + \text{sen}(\theta_1-\theta_2)\dot{\theta}_2^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_2 + 2\omega^2\,\text{sen}\,\theta_1 = 0 \\ \ddot{\theta}_2 - \text{sen}(\theta_1-\theta_2)\dot{\theta}_1^2 + \cos(\theta_1-\theta_2)\ddot{\theta}_1 + \omega^2\,\text{sen}\,\theta_2 = 0 \end{cases}$$

$$\frac{d(\text{pos. angol})}{dt} = (\text{vel. ang}) \qquad \frac{d(\text{vel. ang})}{dt} = (\text{acc. ang})$$

$\left.\begin{array}{l}\theta_1 = z_1 \\ \theta_2 = z_2\end{array}\right)$ posizioni angolari

$\left.\begin{array}{l}\dot{\theta}_1 = z_3 \\ \dot{\theta}_2 = z_4\end{array}\right)$ velocità angolare

$\dot{z}_3 = \ddot{\theta}_1 \qquad \dot{z}_4 = \ddot{\theta}_2$

$\ddot{\theta}_1 = f(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$

$\ddot{\theta}_2 = g($ " $)$

$$\begin{cases} \dot{z}_1 = z_3 \\ \dot{z}_2 = z_4 \\ \dot{z}_3 = \dfrac{\omega^2\left[\text{sen}\,z_2\cos(z_1-z_2) - 2\,\text{sen}\,z_1\right] - \frac{1}{2}\text{sen}\left[2(z_1-z_2)\right]z_3^2 - \text{sen}(z_1-z_2)z_4^2}{2-\cos^2(z_1-z_2)} \\ \dot{z}_4 = \dfrac{2\omega^2\left[\text{sen}\,z_1\cos(z_1-z_2) - \text{sen}\,z_2\right] + 2\,\text{sen}(z_1-z_2)z_3^2 + \frac{1}{2}\text{sen}\left[2(z_1-z_2)\right]z_4^2}{2-\cos^2(z_1-z_2)} \end{cases}$$

$\omega^2 = g/l$

↳ Eqz. differenziali del moto del pendolo doppio

$l = 0{,}5$

$g = 9{,}81$

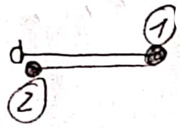
$\omega^2 = g/l$

$\theta_1^{(0)} = \pi/2 \to z_1(0)$

$\theta_2^{(0)} = -\pi/2 \to z_2(0)$

$\dot{\theta}_1(0) = 0 \to z_3(0)$

$\dot{\theta}_2(0) = 0 \to z_4(0)$ → aste ferme



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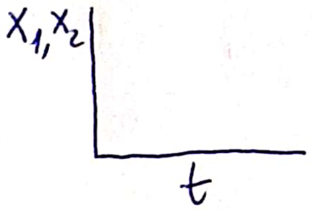

posizione x delle masse

"filmato" pendolo doppio in movimento

traiettoria (nel piano x-y) della massa 2 disegnata dalla massa 2

Sensitività alle condizioni iniziali (traiettorie per due condizioni iniziali "vicine" nel piano x-y)

traiettoria della massa 2 con diversa condizione iniziale