SOLUTION

Politecnico di Milano
Facoltà di Ingegneria

SYSTEMS THEORY (NONLINEAR DYNAMICS)

Prof. Fabio Dercole

November 26th, 2015

LAST and FIRST NAME: ____________________

PERSON CODE or ID NUMBER: ____________________

SIGNATURE: ____________________ Prof. mark: ________

/ 4	/ 3	/ 2	/ 2	/ 6	/ 9	/ 2	/ 2	/3	/33
								homework	total

WARNINGS

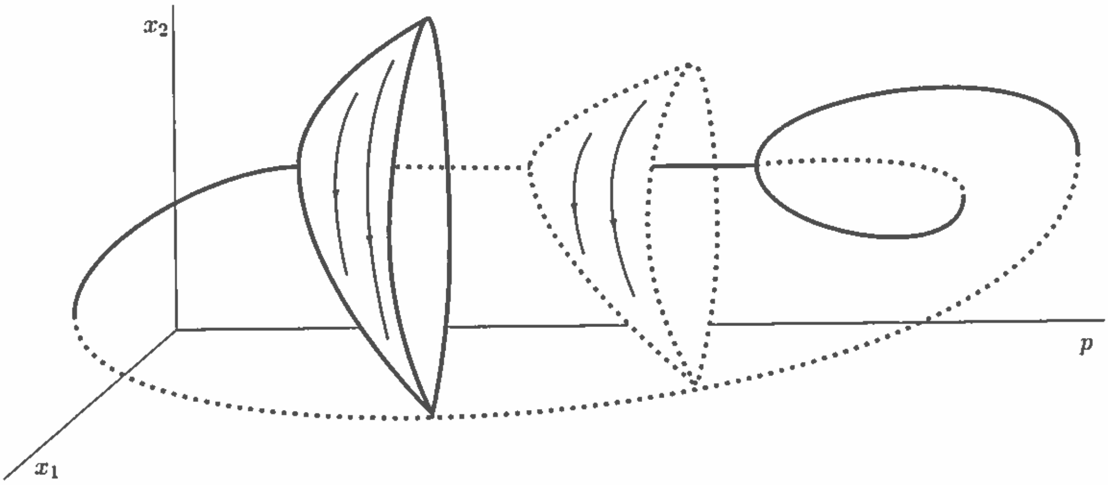
- The grade will be communicated by e-mail in January 2016.
- The corrected exam and homework can be accessed only on:

  Wed. 13th Jan., 5.30 p.m., at Prof. Rinaldi's office (DEIB, 2nd floor, ph. 3563)

- To accept the grade, register for the exam of Feb. 9th with the option "record the grade". The grade will expire if any other action (registration with the option "try the exam" or no registration) is taken.
- Books and notes are forbidden
- Unjustified answers (if not explicitly said) are unacceptable
- Use only the stapled papers
- Order and clarity will be object of evaluation

Problem 1 - 4 points

The stable (solid line) and unstable (dotted line) equilibria and cycles of a second-order continuous-time system are shown along with the change of a model parameter p.



Report the number and type of the bifurcations that you identify in the figure, by indicating the number in the box next to each bifurcation name. No justification is required.

- $\boxed{2}$ Hopf
- $\boxed{0}$ transcritical
- $\boxed{0}$ flip
- $\boxed{1}$ pitchfork
- $\boxed{3}$ seddle-node
- $\boxed{0}$ Neimark-Sacker
- $\boxed{2}$ homoclinic
- $\boxed{0}$ heteroclinic

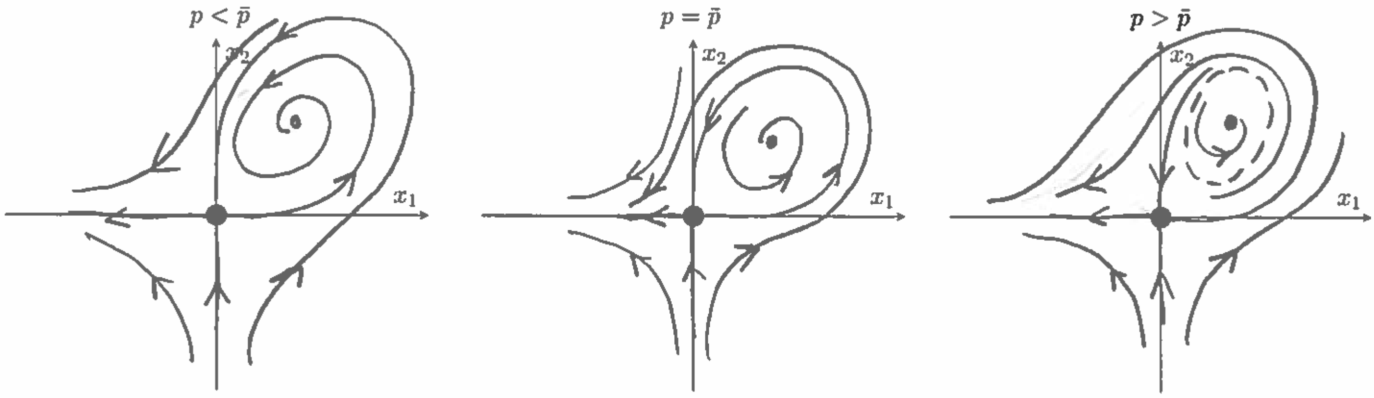
Problem 2 - 3 points

The origin $(x_1, x_2) = (0, 0)$ of a parametric family of second-order continuous-time systems is a saddle equilibrium for all values of a parameter p. For a particular parameter value $\bar{p}$, the origin is involved in a homoclinic bifurcation and the saddle Jacobian at the bifurcation is

$$\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

Draw three possible state portraits, one for $p < \bar{p}$, one for $p = \bar{p}$, and one for $p > \bar{p}$.

SOLUTION



Problem 3 - 2 points

A parametric family of second-order continuous-time systems

$$\dot{x} = f(x, p)$$

is characterized by a unique equilibrium for any value of the parametr p. Underline the true statements without giving any explanation.

- The system cannot undergo a Hopf bifurcation
- The system can ~~a~~ undergo saddle-node bifurcation
- The system cannot undergo a homoclinic bifurcation
- The system cannot have cycles

Warning: correct underlining: 2 points; wrong underlining: -1 point

Problem 4 - 2 points

Describe (in at most 5 lines) a physical system in which you noted (or you think it could be noted) the existence of two (or more) stable equilibria. The system must be different from those discussed in the classes and proposed as exercise.

SOLUTION

roulette

Problem 5 - 6 points

For each of the following statements, say whether the statement is true of false by writing T or F in the corresponding box. If not otherwise indicated, the statements make reference to a continuous-time system. No justification is required.

- $\boxed{\text{F}}$ a flip bifurcation is characterized by a pair of complex conjugate cycle multipliers with unitary modulus
- $\boxed{\text{F}}$ the normal form of the transcritical bifurcation is $\dot{x} = px$
- $\boxed{\text{T}}$ discrete-time systems can be non-reversible
- $\boxed{\text{T}}$ the Shilnikov's theorem is about homoclinic bifurcations
- $\boxed{\text{T}}$ a strange attractor can have two zero Lyapunov exponents
- $\boxed{\text{T}}$ the dimension of a chaotic attractor of a reversible discrete-time systems can be larger than 2
- $\boxed{\text{F}}$ a saddle equilibrium can only have the stable manifold
- $\boxed{\text{T}}$ second-order discrete-time systems can undergo the Neimark-Sacker bifurcation
- $\boxed{\text{F}}$ if the divergence of a third-order system is everywhere negative, the system cannot have chaotic behavior
- $\boxed{\text{F}}$ in third-order systems there cannot be saddle cycles
- $\boxed{\text{F}}$ first-order systems can have heteroclinic bifurcations
- $\boxed{\text{T}}$ first-order systems can have multiple attractors

Warning: correct answer: 0.5 point; wrong answer: -0.5 point

Problem 6 - 9 points

Consider the following parametric family of systems

$$\begin{aligned} \dot{x}_1 &= -x_2 + p \\ \dot{x}_2 &= x_1(x_1-1)(x_1-2) - x_2 \end{aligned}$$

where the parameter p can be positive, zero, or negative.

Consider first the system corresponding to $p = 0$

1. Compute the system's equilibria
2. Analyze the stability of the equilibria and draw the system's trajectories locally to each equilibrium
3. Say whether there can be cycles
4. Draw a possible (global) state portrait of the system

Consider now the family of systems subject to change in the parameter p

5. Determine the bifurcations encountered in the family while varying p and identify the type of bifurcation
6. Draw the branches of system's equilibria in the plane (p, x_1), using solid/dotted lines for stable/unstable equilibria
7. Discuss the presence of an hysteresis

Finally, suppose that p has its own dynamics described by

$$\dot{p} = \varepsilon g(x_1, x_2, p)$$

with very small $\varepsilon > 0$. Say whether it is possible, for a suitable choice of the function g, that the third-order system with state (x_1, x_2, p) has a cycle. In case of a positive answer, propose a choice of g for which the asymptotic behavior of the system is periodic.

SOLUTION

1) $\begin{cases} 0 = -x_2 \\ 0 = x_1(x_1-1)(x_1-2) - x_2 \end{cases}$ $\begin{cases} x_2 = 0 \\ x_1(x_1-1)(x_1-2) = 0 \end{cases} \rightarrow x_1 = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$

$E_0 = (0,0)$, $E_1 = (1,0)$, $E_2 = (2,0)$

2) $J(x_1,x_2) = \begin{bmatrix} 0 & -1 \\ 3x_1^2 - 6x_1 + 2 & -1 \end{bmatrix}$ $\quad \mathrm{tr}J = -1$

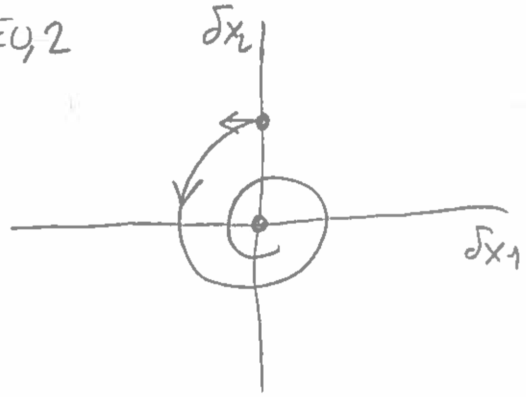
$\det J = 3x_1^2 - 6x_1 + 2 = \begin{cases} 2 \\ -1 \\ 2 \end{cases}$

$\Delta = \mathrm{tr}J^2 - 4\det J = \begin{cases} -7 \\ 5 \\ -7 \end{cases}$

$E_{0,2}$: stable focus, E_1 : saddle

local trajectories

E q 2

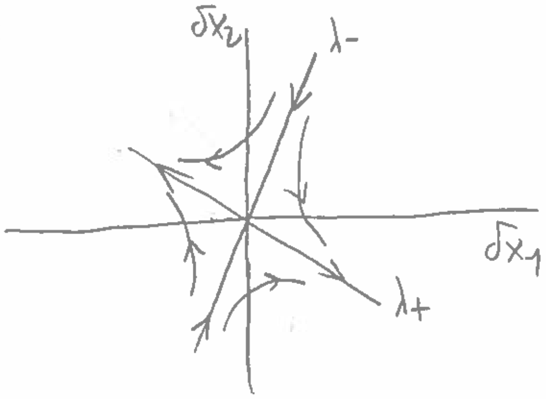


$\delta x(0) = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \rightarrow \dot{\delta x}_1(0) = -1$

$\rightarrow$ counterclockwise rot.

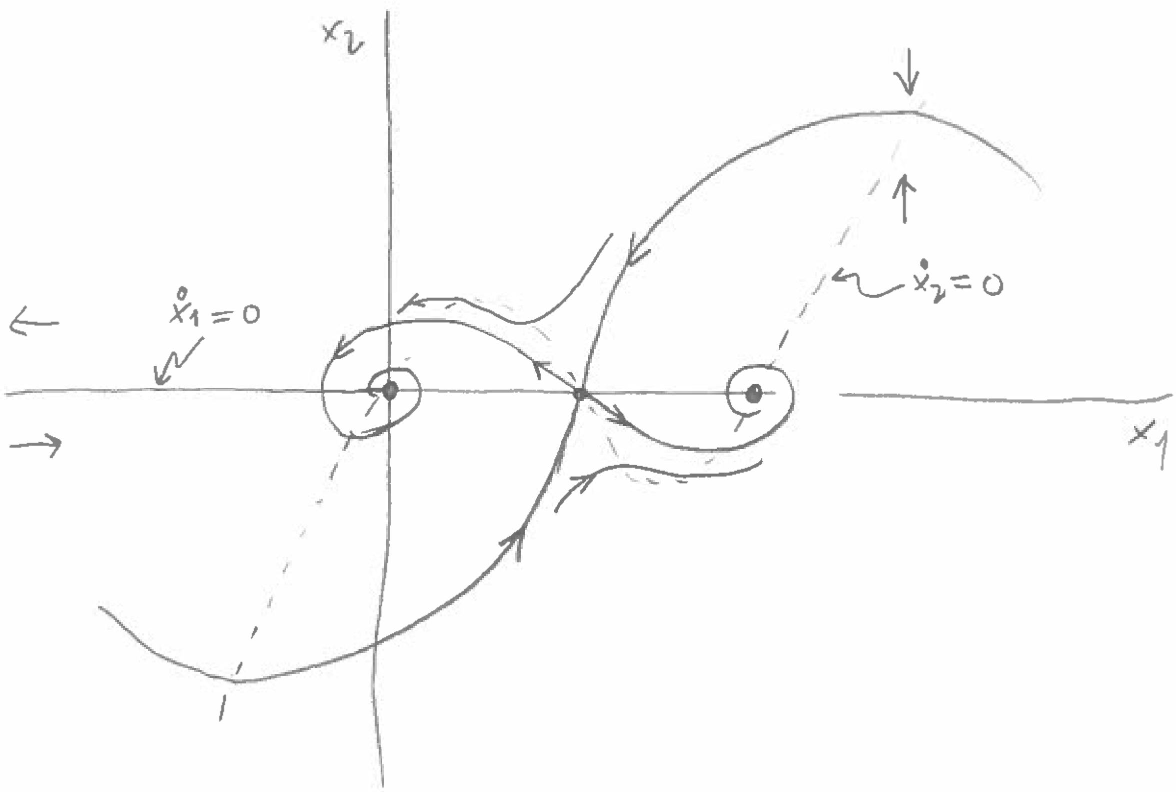
E 1, eigenvalues $\lambda_\pm = \left\{ \frac{-1 \pm \sqrt{5}}{2} \right\}$, eigenvectors $J \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix}$

$\begin{cases} -v_2 = \lambda v_1 \\ \end{cases}$



3) div f = -1 $\rightarrow$ NO cycles

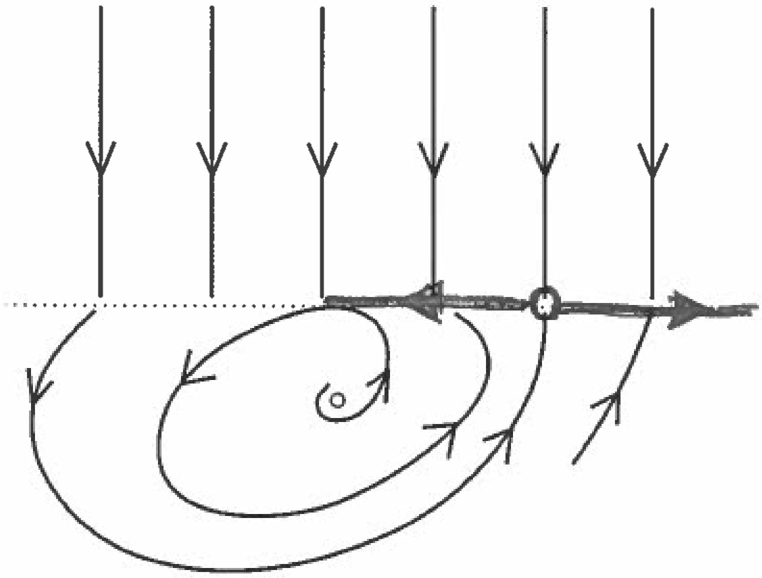
4)



5-7) see back of problem 7

Problem 7 - 2 points

The state portrait of a second-order piecewise-smooth continuous-time system is reported in the figure and shows that the system's equations are discontinuous across the dotted line.



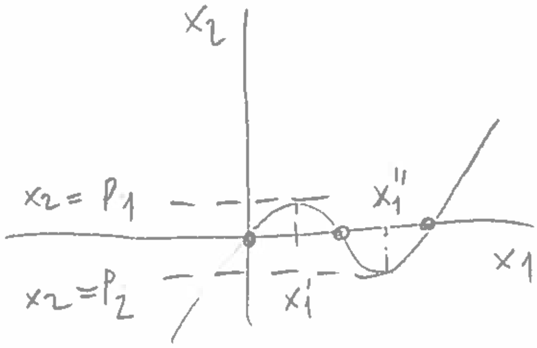
Complete the state portrait by drawing the sliding dynamics on the discontinuity line.

Problem 8 - 2 points

Underline the control technique analyzed in the course for the attitude control of a spacecraft.

- Reaction wheels
- Compressed-air thrusters
- Solid-fuel thrusters
- Magnetic coils

5) $\dot{x}_1$ - nullcline : $x_2 = p$



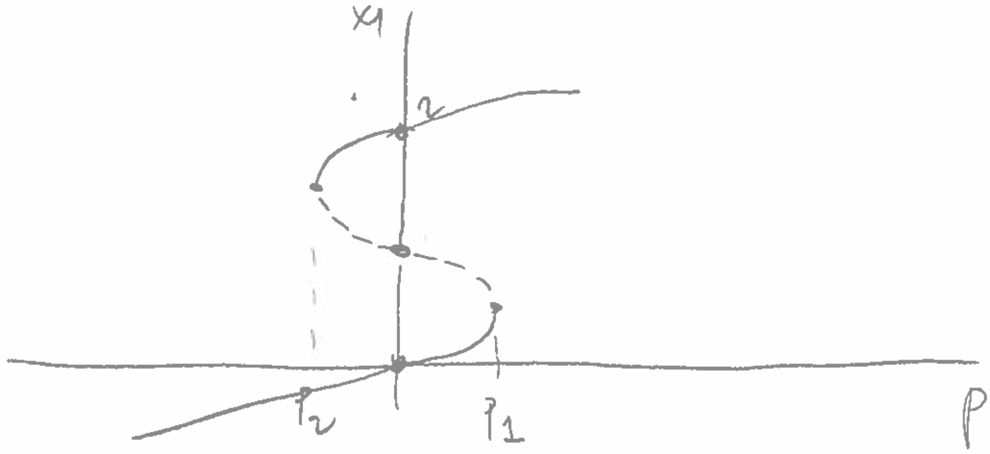
x_1', x_1'' : $\frac{d}{dx_1}\left(x_1(x_1-1)(x_1-2)\right) = 0$

$3x_1^2 - 6x_1 + 2 = 0 \qquad x_1', x_1'' = \frac{3 \pm \sqrt{9-6}}{3} = \frac{3 \pm \sqrt{3}}{3}$

$$\left.\begin{aligned} p_1 &= x_1(x_1-1)(x_1-2)\Big|_{x_1 = x_1'} = \dots \\ p_2 &= x_1(x_1-1)(x_1-2)\Big|_{x_1 = x_1''} = \dots \end{aligned}\right\} \text{saddle-node bifurcations}$$

No other bifurcations are possible since there are no limit cycles.

6)



7) this is the typical hysteretic scenario

8) Yes $g = 1 - x_1$