SOLUTIONS

Politecnico di Milano
Facoltà di Ingegneria

SYSTEMS THEORY (NONLINEAR DYNAMICS)

Prof. Fabio Dercole

February 5th, 2018

LAST and FIRST NAME: ______________________

PERSON CODE or ID NUMBER: ______________________

SIGNATURE: ______________________ Instructor's check: ______

						homework	total
/2	/4	/4	/5	/6	/9	/3	/33

NOTES

- The **exam assessment** can be discussed on **Wed. Feb. 14th, 4.00pm**, in the Instructor's office (DEIB, 2nd floor)
- The use of books and notes is forbidden
- The answers must be justified (unless otherwise stated)
- Use only this set of stapled sheets (use backsides if needed)
- Order and clarity are subject to evaluation

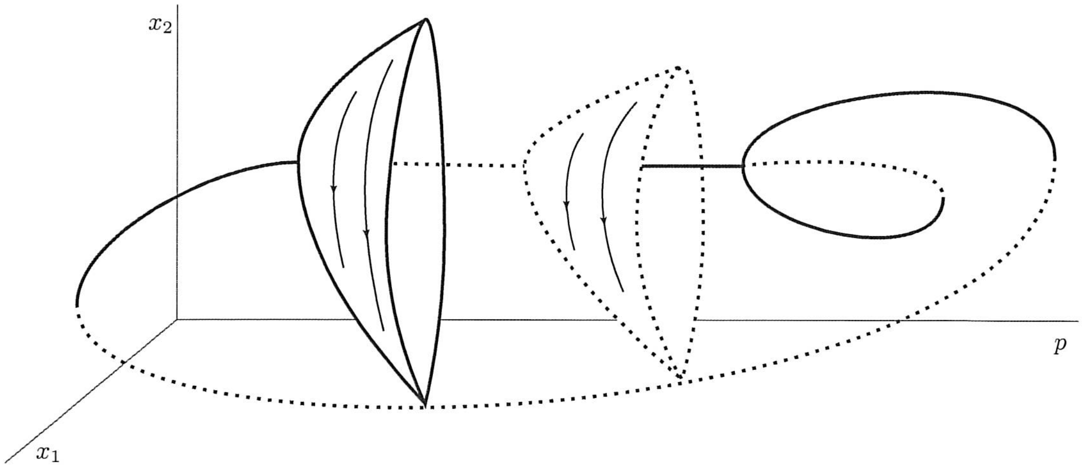
Exercise 1 - 2 points

Describe, in at most 5 liens, an autonomous system (different from those discussed in class and proposed as exercise) characterized by cyclic behavior.

SOLUTION

Exercise 2 - 4 points

The stable (solid line) and unstable (dotted line) equilibria and cycles of a second-order continuous-time system are shown along with the change of a model parameter p.



Report the number and type of the bifurcations that you identify in the figure, by indicating the number in the box next to each bifurcation name. No justification is required.

SOLUTION

2	Hopf
0	transcritical
0	flip
1	pitchfork
3	seddle-node
0	Neimark-Sacker
2	homoclinic
0	heteroclinic

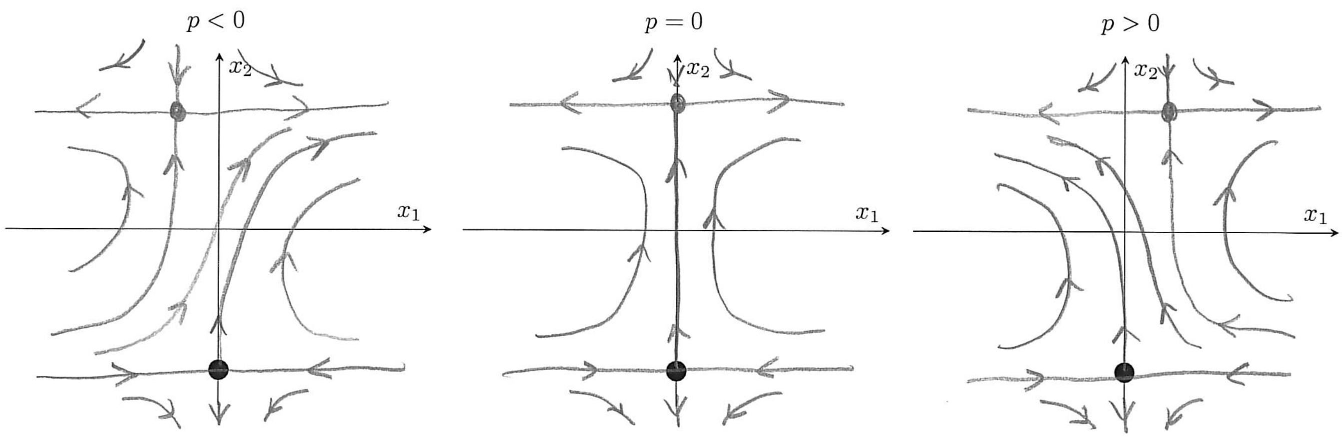
Exercise 3 - 4 points

The points $P_1 = (0, -1)$ (black dot in the axes below) and $P_2 = (p, 1)$ are the only equilibria of a parametric family of second-order continuous-time systems for all values of a single parameter p. For $p = 0$, the two equilibria are involved in an heteroclinic bifurcation and have the following two Jacobian matrices:

$$J_{P_1} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad J_{P_2} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Draw three consistent state portraits for $p < 0$, $p = 0$, and $p > 0$, respectively.

SOLUTION



Exercise 4 - 5 points

For each of the following statements, say whether the statement is true of false by writing T or F in the corresponding box. No justification is required. Unless otherwise specified, the statements make reference to continuous-time systems.

Warning: correct answer: 0.5 point; wrong answer: -0.25 point

F	The Jacobian matrix of an equilibrium of a nonlinear system can be rectangular.
T	A phenomenon of hysteresis requires catastrophic bifurcations.
F	A saddle-node bifurcation can be non-catastrophic.
F	The Hopf bifurcation only occurs in second-order systems.
T	The Neimark-Sacker bifurcation can occur in second-order discrete-time systems.
T	An unstable equilibrium can be attractor.
F	All dynamical systems have attractors.
F	In second-order systems there cannot be saddle cycles.
F	The normal form of the transcritical bifurcation is $\dot{x} = p + x^2$.
T	The Shilnikov's theorem is about homoclinic bifurcations.

Exercise 5 - 6 points
Given the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 - x_2^2 - 1)\\ \dot{x}_2 &= x_2(x_2^2 + 3x_1^2 - 1)\end{aligned}$$

show, by means of the Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$, that the origin of the state space is an asymptotically stable equilibrium and determine a region within its basin of attraction. Then, using the same Lyapunov function, show that the origin is not globally stable.

SOLUTION

$$\dot{V} = 2x_1^2\left(x_1^2 - x_2^2 - 1\right) + 2x_2^2\left(x_2^2 + 3x_1^2 - 1\right) =$$

$$= -2\left(x_1^2 + x_2^2\right)\left(1 - x_1^2 - x_2^2\right)$$

Locally to O: $(x_1, x_2) = (0, 0)$, the factor $(1 - x_1^2 - x_2^2)$ is positive, so that $\dot{V} < 0$ $\Rightarrow$ O is as. stable by the Lyapunov method.

V has closed and "ordered" level curves (circles centered in O), so that by the La Salle method, the largest region around O in which $\dot{V} < 0$ is contained in the basin of attraction of O. This region is $\{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$, i.e., the interior of the unit disk.

For $x(0)$ outside the unit disk $\dot{V}$ is positive, so that the trajectory $x(t)$ must diverge by crossing level curves of V of increasing level. This proves that O is not globally stable.

Exercise 6 - 9 points

Consider the parametric family of second-order systems

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= px_1 - x_2,\end{aligned}$$

where p is a model parameter that can be positive, null, or negative. Address the following questions and provide short justifications to your answers.

1. Determine the equilibria of the system as funtions of the parameter p.
2. Study the stability of the equilibria for $p < 0$ and for $p > 0$.
3. Discuss the existence of cycles.
4. Qualitatively draw the system's nullclines for $p = -1$, $p = 0$, and $p = 1$, indicating the trajectories' direction for each identified region of the plane.
5. Qualitatively draw the state portrait for $p = -1$, $p = 0$, and $p = 1$.
6. Classify the system's bifurcations, specifying the type of each bifurcation and value of p at which it occurs.
7. Qualitatively draw the bifurcation diagram in the space (x_1, x_2, p).

SOLUTION

1) $\begin{cases} x_2 = x_1^3 \\ x_2 = px_1 \end{cases} \rightarrow x_1(p - x_1^2) = 0 \rightarrow \bar{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{x}^{(2,3)} = \begin{bmatrix} \pm\sqrt{p} \\ \pm p\sqrt{p} \end{bmatrix} \quad p > 0$

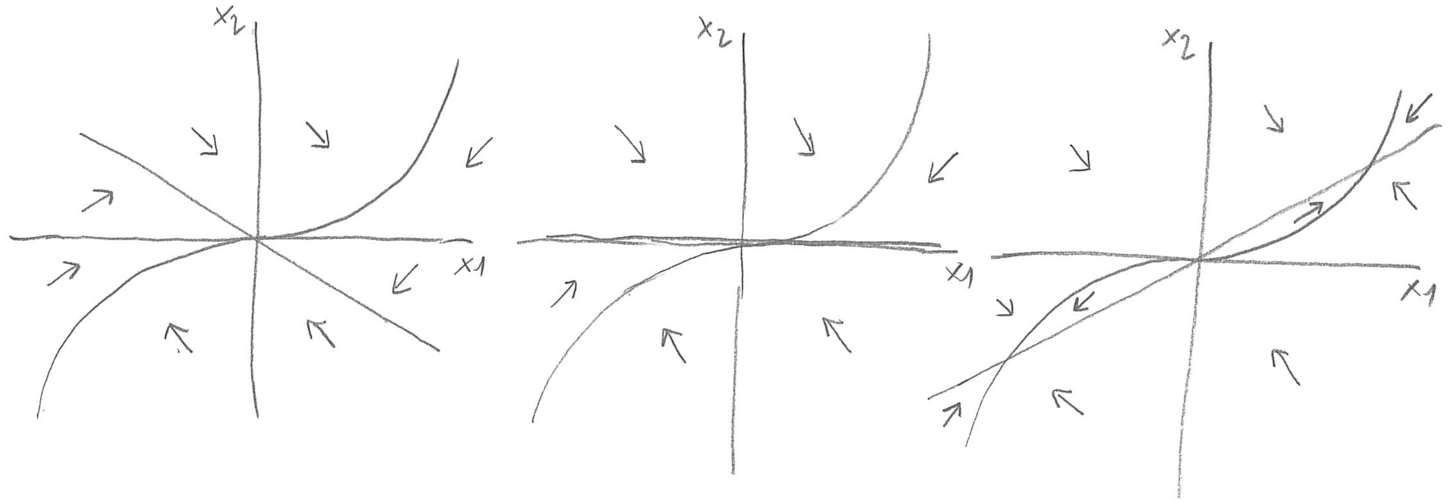
2) $J(x) = \begin{bmatrix} -3x_1^2 & 1 \\ p & -1 \end{bmatrix}$, $J(\bar{x}^{(1)}) = \begin{bmatrix} 0 & 1 \\ p & -1 \end{bmatrix}$ $\quad \det = -p$, $\text{tr} = -1$, $\quad \lambda_{1,2} = \dfrac{-1 \pm \sqrt{1+4p}}{2}$

$\bar{x}^{(1)}$ is $\begin{cases} \text{a stable focus for } p < -1/4 \\ \text{a stable node for } -1/4 \le p < 0 \\ \text{a saddle for } p > 0 \end{cases}$

$J(\bar{x}^{(2,3)}) = \begin{bmatrix} -3p & 1 \\ p & -1 \end{bmatrix}$ $\quad \left.\begin{aligned} &\det = 2p > 0 \text{ for } p > 0 \\ &\text{tr} = -3p - 1 < 0 \text{ for } p > 0 \\ &\text{tr}^2 - 4\det = 9p^2 - 2p + 1 > 0 \ \forall p \end{aligned}\right\} \Rightarrow \bar{x}^{(2,3)}$ are stable nodes for all $p > 1$

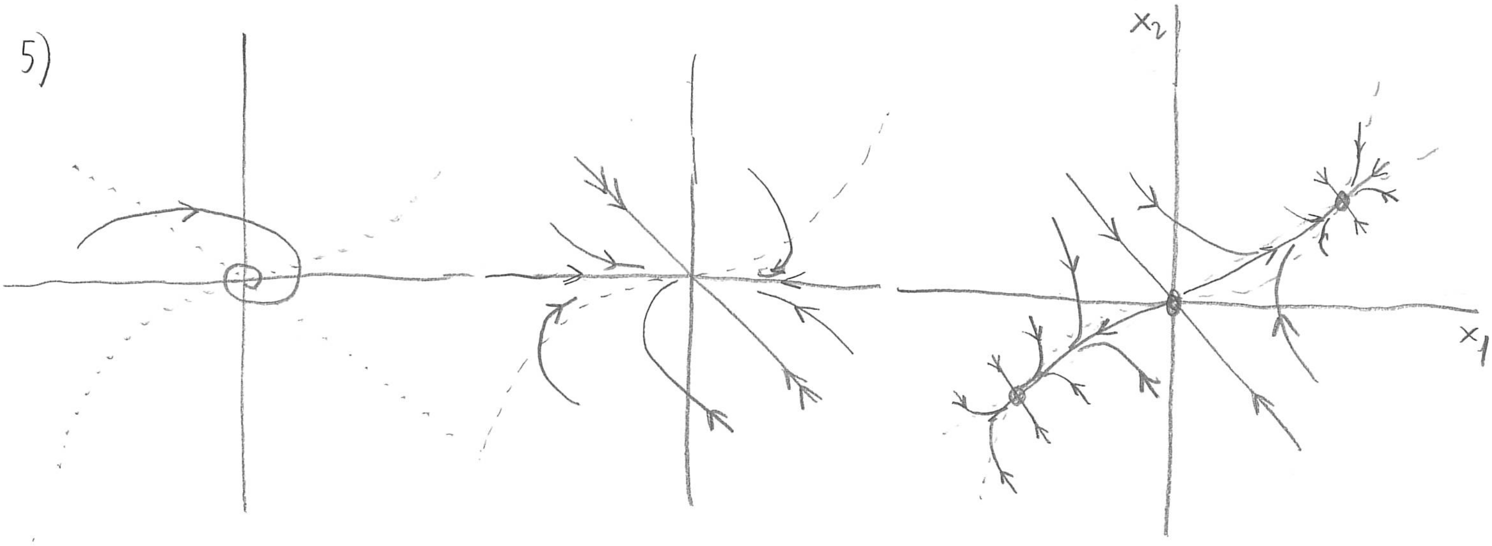
3) $\text{div} f(x) = -3x_1^2 - 1 < 0$ for all $(x_1, x_2) \Rightarrow$ no cycles

4) $\dot{x}_1 \geq 0 : x_2 \geq x_1^3$, $\dot{x}_2 \geq 0 : x_2 \leq p x_1$



$p<0$ $p=0$ $p>0$

5)



$x_2 = 0 \Rightarrow \dot{x}_1 = -x_1^3 \Rightarrow x = 0$ as. stab

6) $p=0$ super-critical pitchfork

7)