SOLUTION

Politecnico di Milano
Facoltà di Ingegneria

SYSTEMS THEORY (NONLINEAR DYNAMICS)

Prof. Fabio Dercole

January 15th, 2018

LAST and FIRST NAME: ______________________________

PERSON CODE or ID NUMBER: ______________________________

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SIGNATURE: ______________________________ Instructor's check: ______

						homework	total
/2	/6	/4	/5	/4	/9	/3	/33

NOTES

- **Graded exams** can be discussed on **Jan. 25th, 4.00pm**, in the Instructor's office (DEIB, 2nd floor)
- The use of books and notes is forbidden
- The answers must be justified (unless otherwise stated)
- Use only this set of stapled sheets (use backsides if needed)
- Order and clarity are subject to evaluation

Exercise 1 - 2 points

Describe, in at most 5 lines, a physical system that can behave on a quasi-periodic regime. The system must be different from those discussed in class and proposed as exercise.

SOLUTION

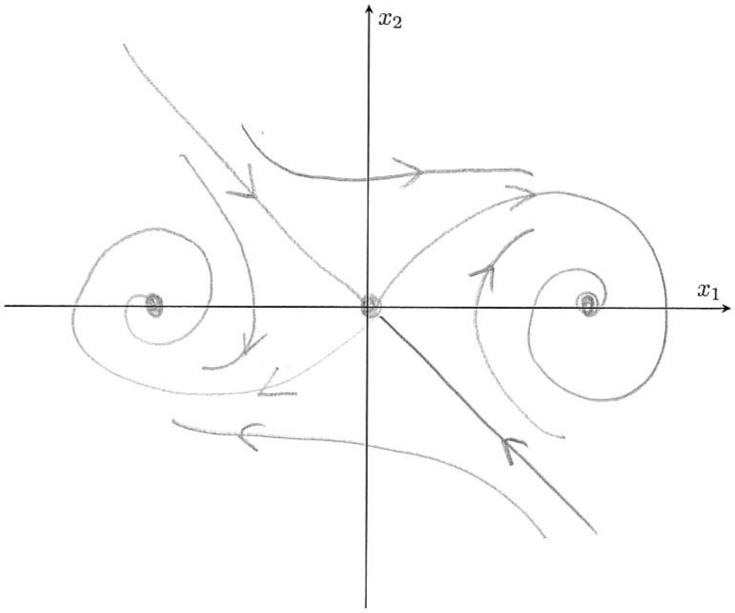
Exercise 2 - 6 points

a. Draw the state portrait of a continuous-time, second-order system characterized by 3 equilibria and no cycles.

b. Say whether a linear system can have a state portrait of the kind of the one drawn. Provide a short justification to your answer.

SOLUTION

a)



b) No, linear systems can have no, one, or infinitely many equilibria

Exercise 3 - 4 points

Assuming $\bar{x}$ to be an equilibrium of the system $\dot{x} = f(x)$ and V and $-(\partial V/\partial x)f$ to be positive-definite functions at $\bar{x}$, then

- $\bar{x}$ is stable, but not asymptotically stable
- $\bar{x}$ is globally stable
- $\bar{x}$ is unstable
- $\bar{x}$ is asymptotically stable.

Underline the true conclusion and provide a short justification to your answer.

SOLUTION

By the Lyapunov method.

Exercise 4 - 5 points

For each of the following statements, say whether the statement is true of false by writing T or F in the corresponding box. No justification is required. The statements make reference to continuous-time systems.

Warning: correct answer: 0.5 point; wrong answer: -0.25 point

- [T] A cycle of a second-order system can undergo a fold bifurcation.
- [F] If the divergence of a third-order system is everywhere negative, cycles cannot exist.
- [T] The normal form of the transcritical bifurcation has dimension 1.
- [F] The equilibria involved in a saddle-node bifurcation must necessarily be a saddle and a node.
- [F] A linear system cannot have more than one equilibrium.
- [T] A linear system can have cycles.
- [T] The equilibrium of an asymptotically stable linear system is also globally stable.
- [F] The sum of the Lyapunov exponents of an attractor can be positive.
- [F] The Jacobian's trace of a third-order system evaluated at an equilibrium undergoing a Hopf bifurcation is zero.
- [T] The Jacobian's determinant of a third-order system evaluated at an equilibrium undergoing a saddle-node bifurcation is zero.

Exercise 5 - 4 points

A second-order system of the kind $\dot{x} = f(x)$ has a single equilibrium at the origin and the Jacobian at $(0, 0)$ is

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Can the system have cycles? Answer Yes or No and justify your answer.

SOLUTION(sono ammesse discussioni grafiche)

No. Cycles in continuous-time, second-order systems must contain a non-saddle equilibrium (node or focus).

Exercise 6 - 9 points

The dynamics of an electric circuit is described by the following two equations

$$\begin{aligned}\dot{x}_1 &= x_2 - Rx_1\\ \dot{x}_2 &= -x_1 + \tfrac{1}{2}x_2 - x_2^3,\end{aligned}$$

where x_1 is a current, x_2 is a voltage, and R is a constant resistor.

a. Determine the equilibria of the system as functions of $R > 0$.

b. Study the stability of the equilibria found at the previous point.

c. Discuss the existence of cycles.

d. Qualitatively draw the state portrait for $R = 3$ and briefly discuss the system's behavior.

e. Classify the system's bifurcations, specifying the values of R at which they occur.

f. Qualitatively draw the bifurcation diagram in the space (R, x_1, x_2).

SOLUTION

a) $\begin{cases}\dot{x}_1 = 0\\ \dot{x}_2 = 0\end{cases}$ $\quad\begin{cases}x_2 = Rx_1\\ -x_1 + \frac{1}{2}Rx_1 - (Rx_1)^3 = 0\end{cases}$

$\bar{x}^{(1)} = \begin{bmatrix}0\\0\end{bmatrix} = \vartheta$

$-1 + \frac{1}{2}R - R^3\bar{x}_1^2 = 0 \rightarrow \bar{x}_1 = \pm\frac{1}{R}\sqrt{\frac{1}{2} - \frac{1}{R}} \rightarrow \bar{x}^{(2,3)} = \begin{bmatrix}\bar{x}_1\\ R\bar{x}_1\end{bmatrix}$

$R \leq 2$: the system has only one eq. $\bar{x}^{(1)}$

$R > 2$: the system has two more equilibria $\bar{x}^{(2)}$ and $\bar{x}^{(3)}$

b) $J(x) = \begin{bmatrix}-R & 1\\ -1 & \frac{1}{2} - 3x_2^2\end{bmatrix}$

$J(\bar{x}^{(1)}) = \begin{bmatrix}-R & 1\\ -1 & \frac{1}{2}\end{bmatrix}$ $\quad$ $tr = -R + \frac{1}{2} < 0$ for $R > \frac{1}{2}$

$\qquad\qquad\qquad\qquad\quad$ $det = -\frac{R}{2} + 1 > 0$ for $R < 2$

$\Delta = tr^2 - 4det = R^2 + R - \frac{15}{4} > 0$ for $R > \frac{3}{2}$ $\left(R_{1,2} = \frac{-1 \pm \sqrt{1+15}}{2} \begin{array}{l} 3/2 \\ -5/2\end{array}\right)$

$\bar{x}^{(1)}$ is $\begin{cases}\text{an unstable focus for } 0 \leq R < 1/2\\ \text{a stable focus for } \frac{1}{2} < R < 3/2\\ \text{a stable node for } 3/2 \leq R < 2\\ \text{a saddle for } R > 2\end{cases}$

$$J(\bar{x}^{(3,3)}) = \begin{bmatrix} -R & 1 \\ -1 & \frac{3}{R}-1 \end{bmatrix} \qquad \left(3\bar{x}_2^2 = 3R^2\bar{x}_1^2 = \tfrac{3}{2} - \tfrac{3}{R}\right)$$

$tr = -R + \frac{3}{R} - 1 < 0$, $R^2 + R - 3 > 0$ for $R > \frac{-1+\sqrt{13}}{2} \cong 1{,}3$

$det = -3 + R + 1 > 0$ for $R > 2$

$\Delta = tr^2 - 4\,det > 0$ for $R > 2$ (at $R=2$ $tr = -\frac{3}{2}$ and $det = 0$, then for $R>2$ tr^2 grows with R more than $4\,det$ - optional to show)

$\bar{x}^{(2,3)}$ are stable nodes for $R > 2$

c) $div\, f = -R + \frac{1}{2} - 3x_2^2$

$R > \frac{1}{2} \rightarrow div f < 0 \rightarrow$ no cycles

$R = \frac{1}{2} \rightarrow div f < 0$ for $x_2 \neq 0$, $div f = 0$ for $x_2 = 0$ (set of zero measure) $\rightarrow$ no cycles

$R < \frac{1}{2} \rightarrow$ there can be cycles

d) Linearization at $\bar{x}^{(2,3)}$

$$J = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \qquad \lambda_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2} \cong \begin{cases} -0{,}38 \ (\lambda_1) \\ -2{,}61 \ (\lambda_2) \end{cases} \text{(eigenvalues)}$$

$$Jv = \lambda v \quad \begin{cases} \dots \\ -v_1 = \lambda v_2 \end{cases} \rightarrow v_2 = -\frac{1}{\lambda_{1,2}} v_1 \quad \text{(eigenvectors)}$$

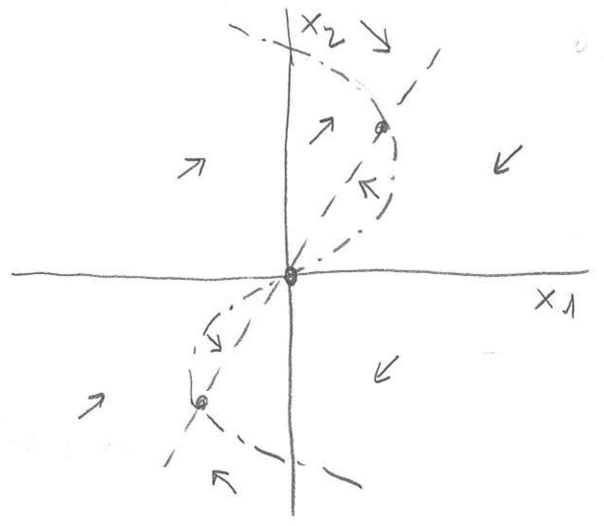
$v_2 \cong 2{,}63\, v_1$ (λ_1 - dominant)

$v_2 \cong 0{,}38\, v_1$ (λ_2 - fastest)

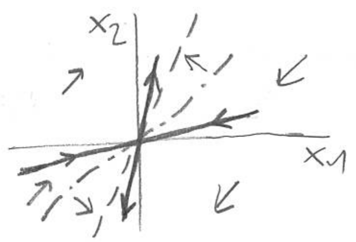
Nullclines

$\dot{x}_1 > 0$, $x_2 > 3x_1$ (- -)

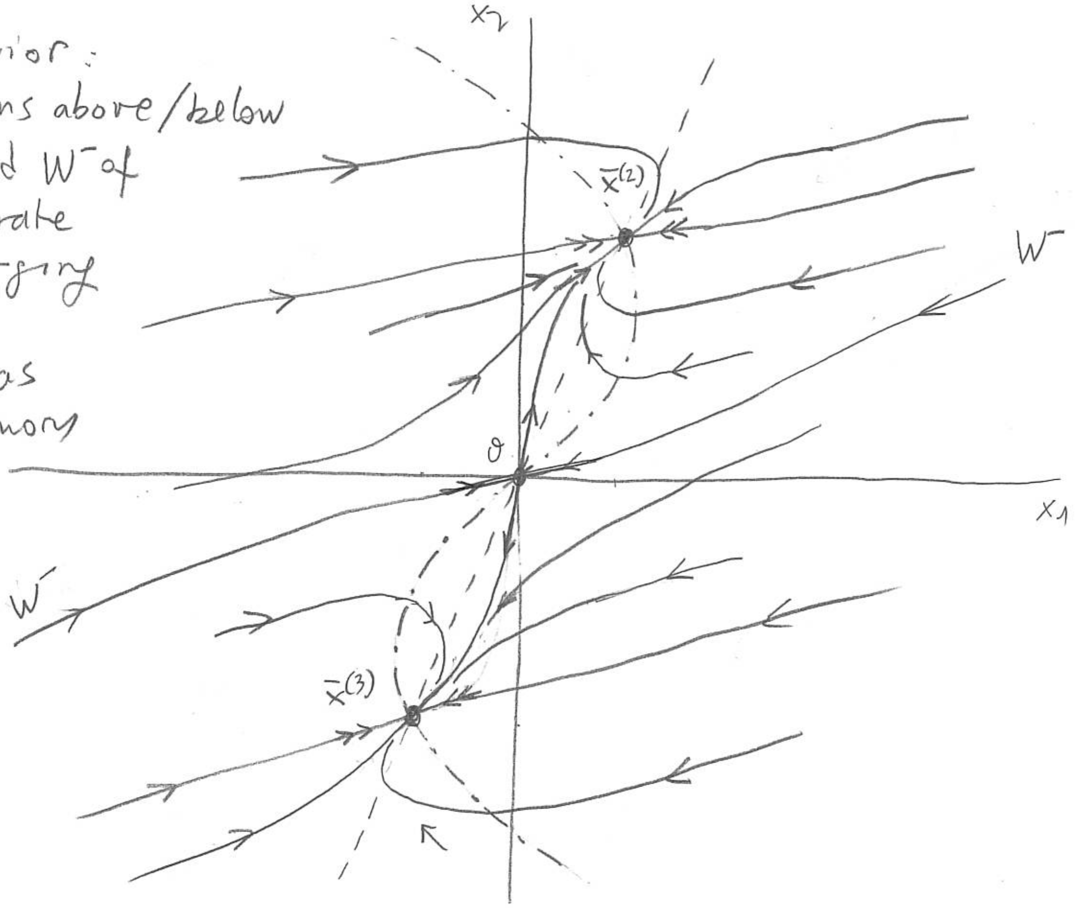
$\dot{x}_2 > 0$, $x_1 < \frac{1}{2}x_2 - x_2^3$ (-.-)



the linearitation at $\bar{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not necessary. According to the nullclines plot, the stable and unstable manifolds of the saddle $\bar{x}^{(1)}$ must be (locally) as shown.



System's behavior:
the initial conditions above/below the stable manifold W^- of the saddle O generate trajectories converging to $\bar{x}^{(2)}/\bar{x}^{(3)}$.
The circuit works as a element of memory (flip-flop)



e)

the Hopf is supercritical because the cycle is present for $R < \frac{1}{2}$ (see point c) and the origin is an unstable focus, so that the cycle is stable

f)