SOLUTION

Politecnico di Milano
Facoltà di Ingegneria

SYSTEM'S THEORY (NONLINEAR DYNAMICS)

Prof. Fabio Dercole

November 28th, 2014

SURNAME and NAME:

PERSON CODE: ID NUMBER:

SIGNATURE: ____________________ ID check: ______

4	6	2	5
3	2	8	1

Evaluation:

total	homework	
31	3	

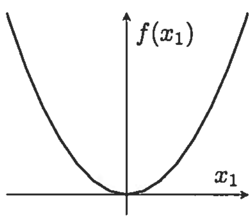
The discussion of the evaluation will be possible only on
Wed. Dec. 11th, 6.00 pm, at the Professor's office (DEIB, 2 floor, tel. 3484)

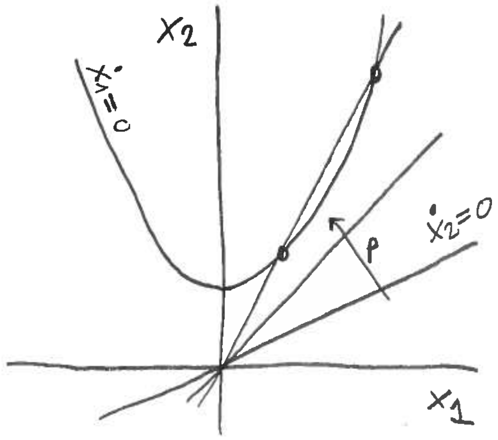
- Consulting books and notes is forbidden
- Unjustified answers (if not explicitly required) will not be evaluated
- Answers must be written exclusively on the present stapled booklet
- Order and clarity will be subject to evaluation

Problem 1 - 4 points

Given the system

$$\begin{aligned}\dot{x}_1 &= f(x_1) - x_2 + 1 \\ \dot{x}_2 &= -px_1 + x_2\end{aligned}$$





with positive parameter p and function $f(x_1)$ shown in the figure (note that $f''(x_1) > 0$ for any x_1), answer the following YES/NO questions, justifying your answers within the provided spaces (Suggestion: use graphical methods).

Do transcritical bifurcations occur? Answer: NO

Justification: The analysis of the nullclines show that the system can have either 0 or 2 equilibria, depending on p. When the equilibria collide, while decreasing p, they disappear, so that the bifurcation is not transcritical

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Do saddle-node bifurcations occur? Answer: Yes

Justification: shown by the nullclines

Do pitchfork bifurcations occur? Answer: No

Justification: there cannot be 3 equilibria (see first point)

Do Hopf bifurcations occur? Answer: NO

Justification: the system's Jacobian at the equilibrium $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is

$$J(\bar{x}) = \begin{bmatrix} f'(\bar{x}_1) & -1 \\ -p & 1 \end{bmatrix}$$, so that the Hopf condition is $f'(\bar{x}_1) + 1 = 0$.

However the condition cannot be satisfied, because $\bar{x}_1$ must be positive (see the nullclines) and $f'(\bar{x}_1) > 0$ for $\bar{x}_1 > 0$.

Problem 2 - 6 points

Given the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 - x_2^2 - 1)\\ \dot{x}_2 &= x_2(x_2^2 + 3x_1^2 - 1)\end{aligned}$$

show, by means of a Lyapunov function, that the origin of the state space is an asymptotically stable equilibrium and determine a region within its basin of attraction. Then, using the same Lyapunov function, show that the origin is not globally stable.

Solution

Try the Lyapunov func. $V(x_1, x_2) = x_1^2 + x_2^2$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \cdots = -2(x_1^2+x_2^2)\left(1-(x_1^2+x_2^2)\right)$$

Locally to $(x_1, x_2) = (0,0)$, $\dot{V} < 0$, that proves the asymptotic stability of the origin

V has closed and "ordered" level curves (circles centered in $(0,0)$), so that, by the La Salle method, the largest region around $(0,0)$ in which $\dot{V} < 0$ is contained in the basin of attraction of $(0,0)$. This region is $\{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$, i.e., the unit disc.

$(0,0)$ is not globally stable. In fact $\dot{V} > 0$ outside the unit disc and V, as already noted, has closed and "ordered" level curves. Thus, starting from an initial condition $x(0)$ outside the unit disc, the system's trajectory must diverge.

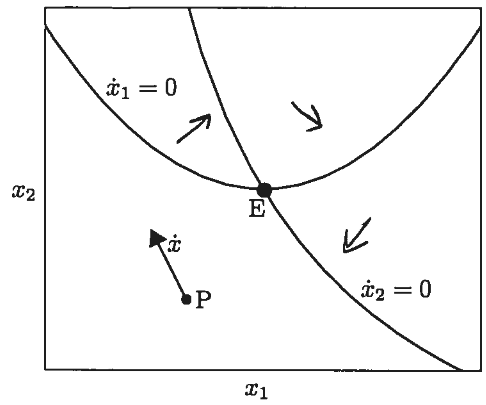
Problem 3 - 3 points

Underline the (only one) statement that holds true, without giving any explanation.

- The matrix A of a linear system $\dot{x} = Ax$ can be rectangular.
- A saddle-node bifurcation can be non-catastrophic.
- A second-order continuous-time system can have only two equilibria both stable.
- The Neimark-Sacker bifurcation can occur in first-order discrete-time systems.
- The Hopf bifurcation only occurs in second-order systems.
- A Hopf bifurcation can be non-catastrophic.
- An asymptotically stable equilibrium can also be unstable.
- There are no systems without attractors.
- A system with a single equilibrium can undergo a pitchfork bifurcation.
- In fourth-order systems there cannot be saddle cycles.

Problem 4 - 5 points

In a second-order continuous-time system, the null-clines intersect as in figure.



Knowing the vector $\dot{x}$ (tangent to the trajectory) at point P (see the figure), say whether the equilibrium E is of focus, node, or saddle type. Justify your answer within the provided space.

Solution

Equilibrium E is of ....FOCUS.... type

Justification: the arrows in the figure only allow a spiraling behavior
Note that the stability of E cannot be discussed based on the available information.

Problem 5 - 3 points

Describe, in at most 5 liens, a system (different from those discussed in class) characterized by cyclic behavior.

Solution

A neon lamp near to breakdown is flashing.

Problem 6 - 2 points

With reference to continuous-time systems ($\dot{x} = f(x)$) and to equilibria bifurcation, say whether the eigenvalue that is critical at the bifurcation is imaginary, positive, negative, null, or unitary.

Solution (with no justification. Just write imaginary, positive, negative, null, or unitary)

transcritical: null

saddle-node: null

pitchfork: null

Hopf: imaginary

Problem 7 - 8 points

Given the system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -x_1^2 + x_2 \end{aligned}$$

answer the following points, justifying your answers within the provided spaces:

1. Does the system have equilibria different from the origin of the state space?
2. Is the origin a stable node, an unstable node, a stable focus, an unstable focus, or a saddle?
3. Are there cycles in the system?
4. Sketch the state portrait locally to the origin.
5. Draw the null-clines globally in the state plane.
6. Sketch the full state portrait of the system (A rather qualitative answer is acceptable).

Solution

1. Does the system have equilibria different from the origin of the state space? Yes

Justification:

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 = 0 \\ \dot{x}_2 = -x_1^2 + x_2 = 0 \end{cases} \qquad \begin{cases} x_2 = 3x_1 \\ x_1(-x_1 + 3) = 0 \end{cases} \Rightarrow$$

2 equilibria
$E_1 = (0,0)$
$E_2 = (3,9)$

2. Is the origin a stable node, an unstable node, a stable focus, an unstable focus, or a saddle? Saddle

Justification:

$$J(0,0) = \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix} \qquad \lambda_1 = -3, \lambda_2 = 1$$

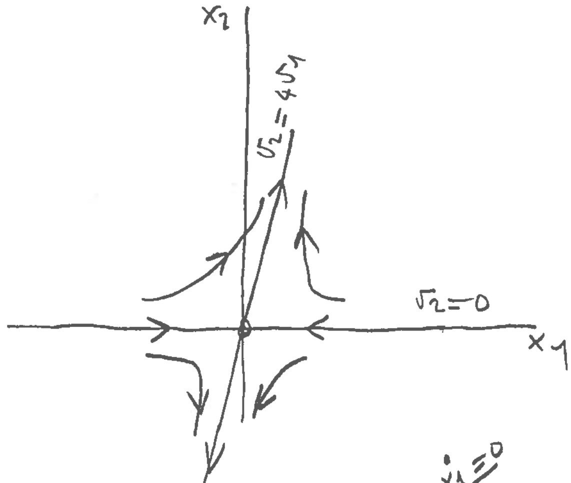
3. Are there cycles in the system? NO

Justification:

$\operatorname{div} f = -3+1 = -2$ does not change sign in the state plane

4. Sketch the state portrait locally to the origin.

Justification:



eigenvectors at $(0,0)$

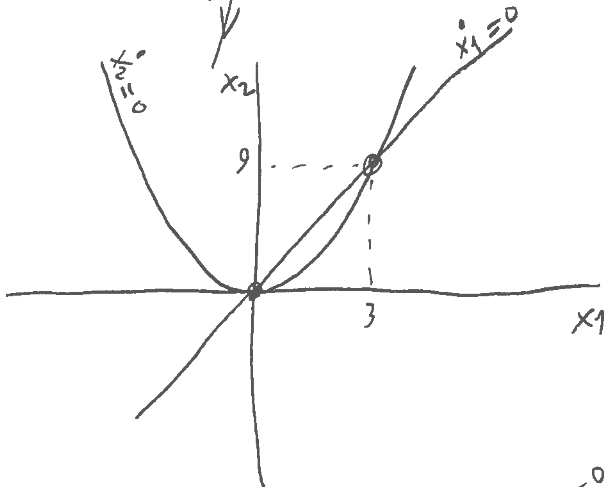
$$\begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_{1,2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\lambda_1 = -3 \Rightarrow v_2 = -3v_2 \Rightarrow v_2 = 0$

$\lambda_2 = 1 \Rightarrow -3v_1 + v_2 = v_1 \Rightarrow v_2 = 4v_1$

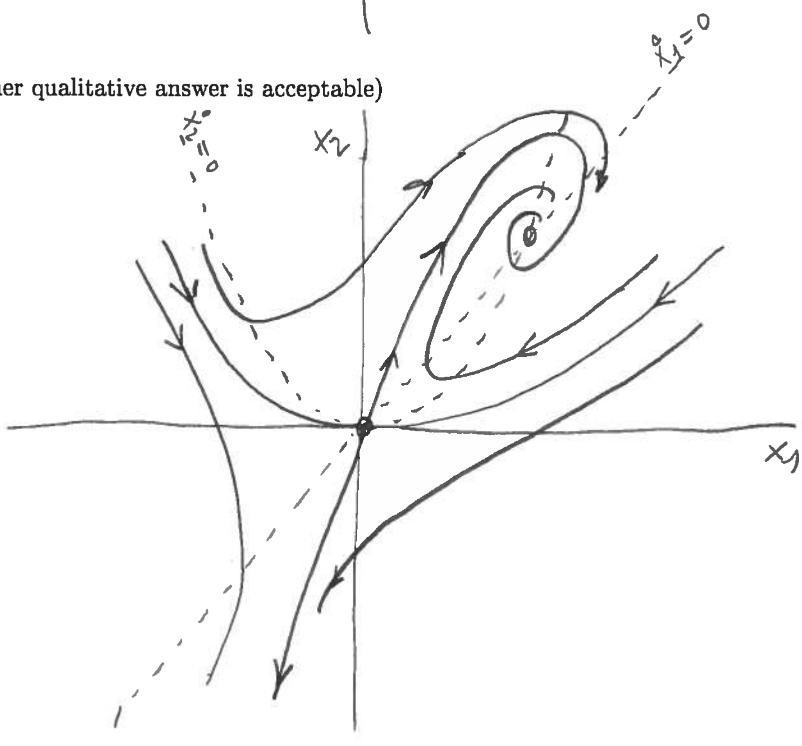
5. Draw the null-clines globally in the state plane.

Justification:



6. Sketch the full state portrait of the system (A rather qualitative answer is acceptable)

Justification:



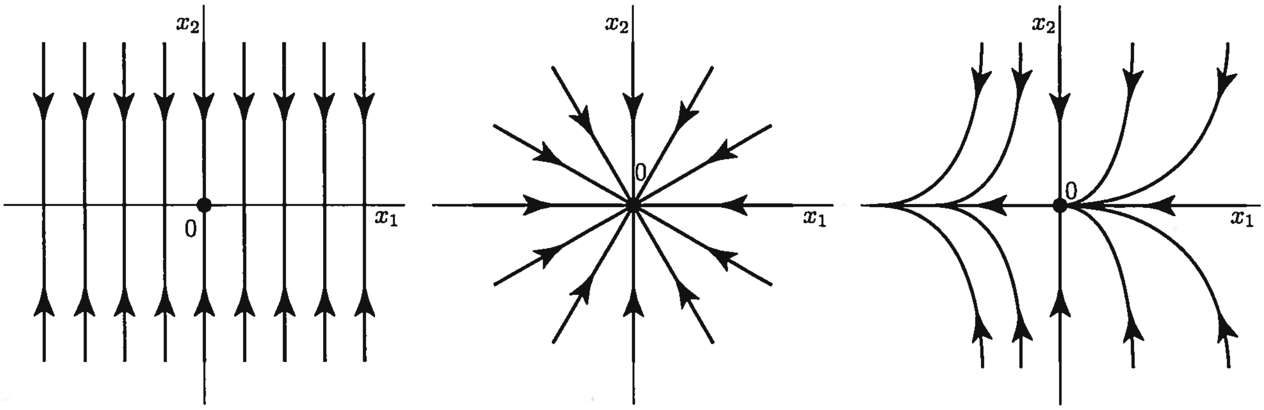
Problem 8 - 1 point

Suggestion: Solve this (nontrivial) problem only if time is left after solving Problems 1–7.

Given the second-order system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= -x_2\end{aligned}$$

where function f_1 is unspecified, define three expressions for f_1 such that the state portrait of the system is that of figure (a), (b), (c), respectively.



(a) (b) (c)

Solution (answer in the provided spaces)

(a) $f_1(x_1, x_2) = 0$

Justification: x_1 does not change along the system's trajectories. The eq. for $\dot{x}_2$ is consistent with fig. (a)

(b) $f_1(x_1, x_2) = -x_1$

Justification: fig (b) is consistent with a linear star (2 eigenvalues at -1 generating 2 independent eigenvectors)

(c) $f_1(x_1, x_2) = -x_1^2$

Justification: x_1 must always decrease, except if $x_1 = 0$

The eq. for $\dot{x}_2$ is consistent with fig (c).

The system is nonlinear with a saddle-node equilibrium at $(0,0)$.