ABCs of nonsmooth systems

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the bouncing ball

$$\dot{x} = v$$
$$\dot{v} = -g$$

if $x = 0$ then $v \mapsto -\rho v$

coefficient of restitution

A sliding mass with Coulomb friction

$$\dot{x} = v$$
$$m\dot{v} = f_g \sin(\theta) + C$$

Formalism 1: hybrid system

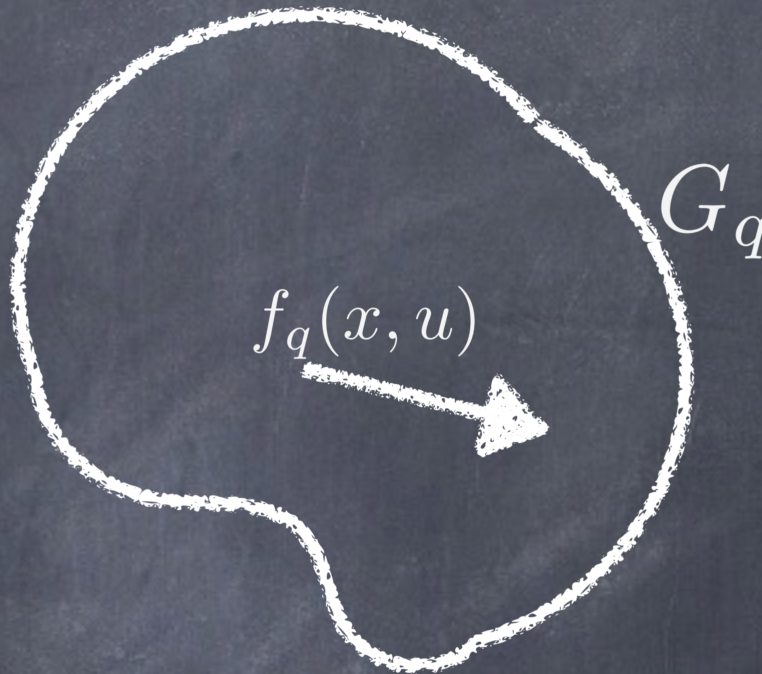
continuous state

$$\dot{x} = f_q(x, u)$$ ← input

"discrete state" or "mode"

if $x \in G_q$ then $\begin{array}{l} x \mapsto x' \\ q \mapsto q' \end{array}$ ← reset map

"discontinuity set" or "guard"



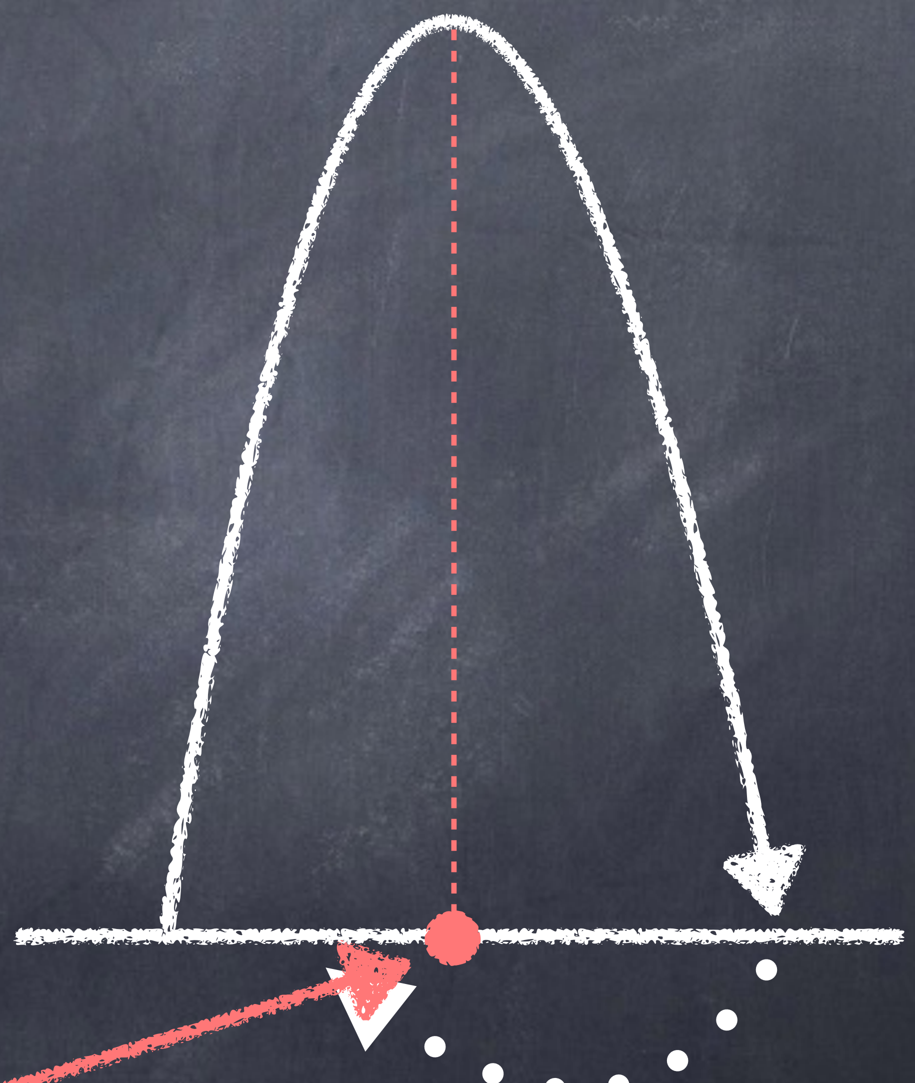
the state of the system is (x,q)

Example: bouncing ball as a hybrid system

$$f_1(x,v) := \begin{pmatrix} v \\ -g \end{pmatrix}$$

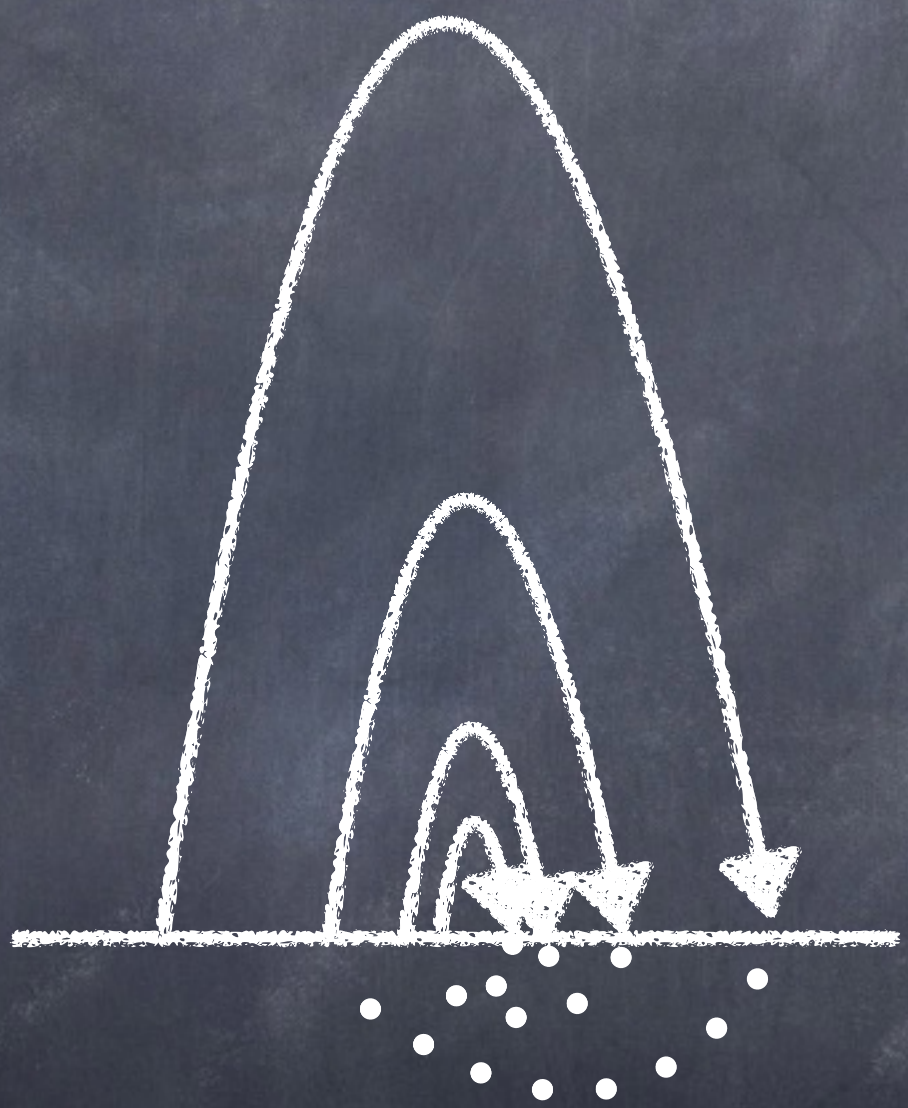
$$G_1 := \{(x,v) : x = 0\}$$

$$\text{if } x \in G_1 \text{ then } \; x \mapsto x, v \mapsto -\rho v$$

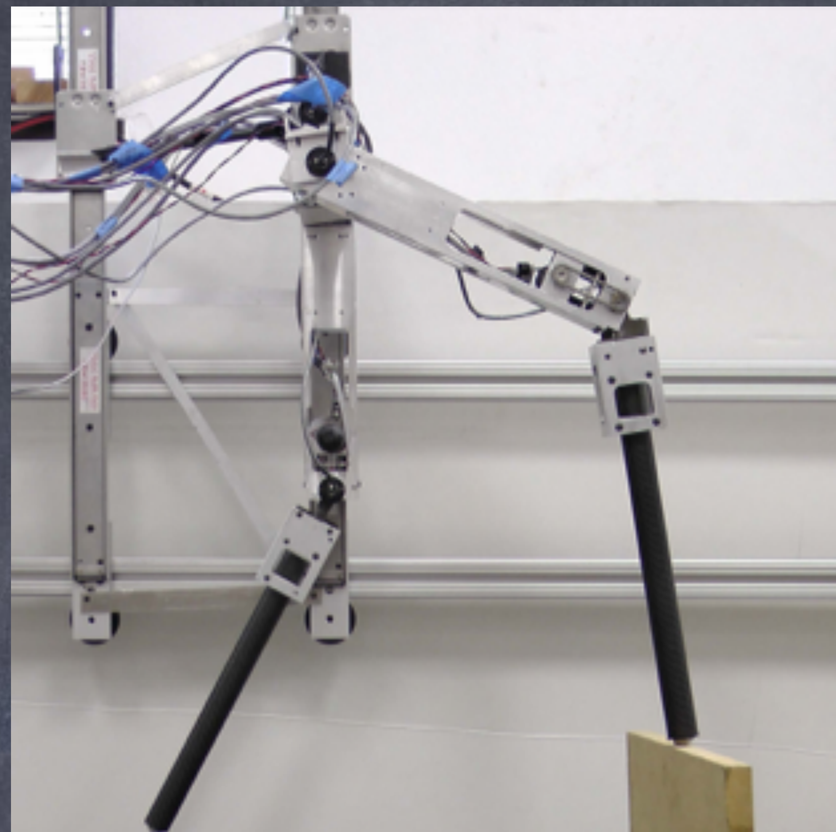


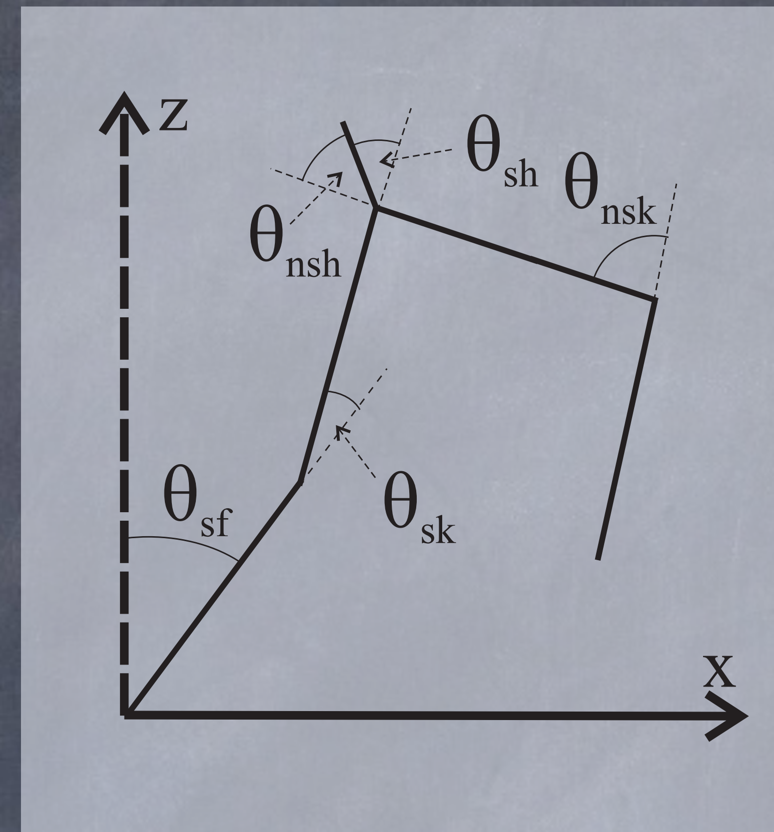
notice that the fixed point of the reset map coincides with a tangent point of the vector field

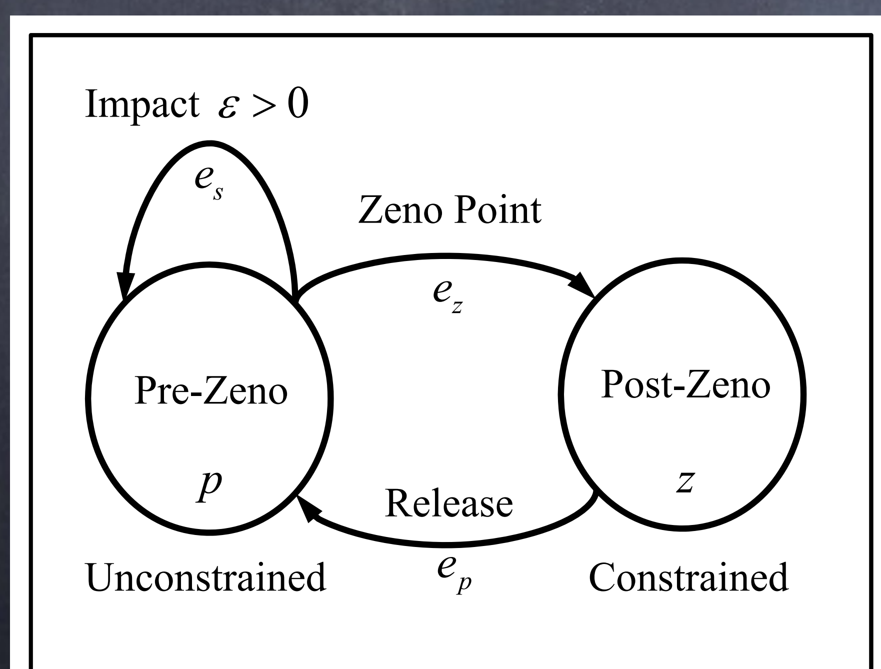
Chattering in a hybrid system (or Zeno effect)

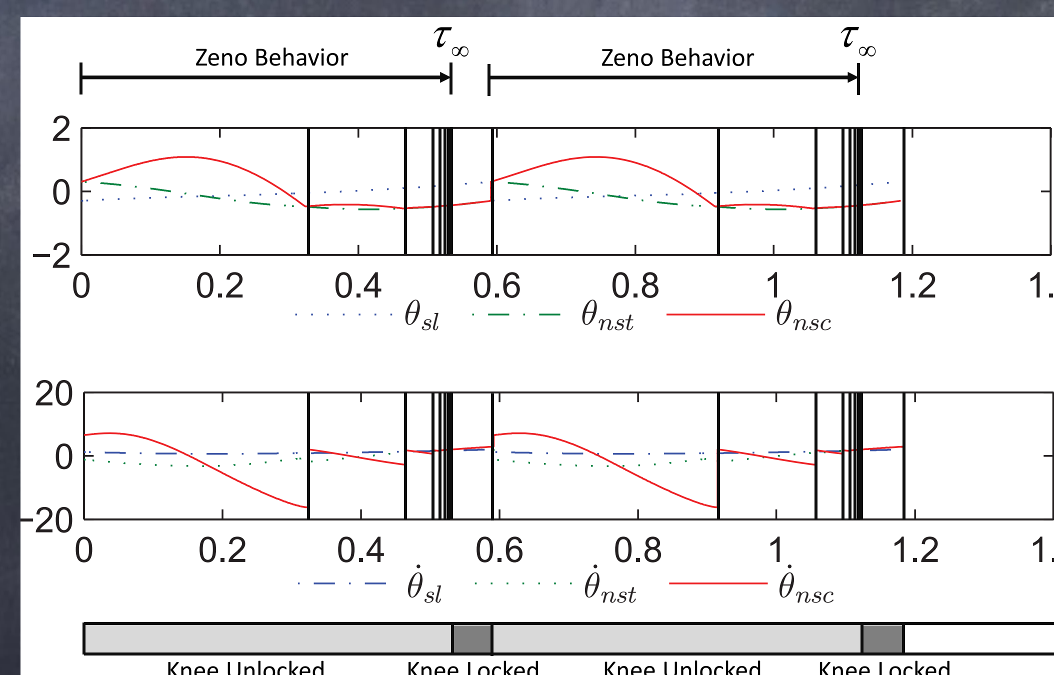


an infinite number of switches in a finite time







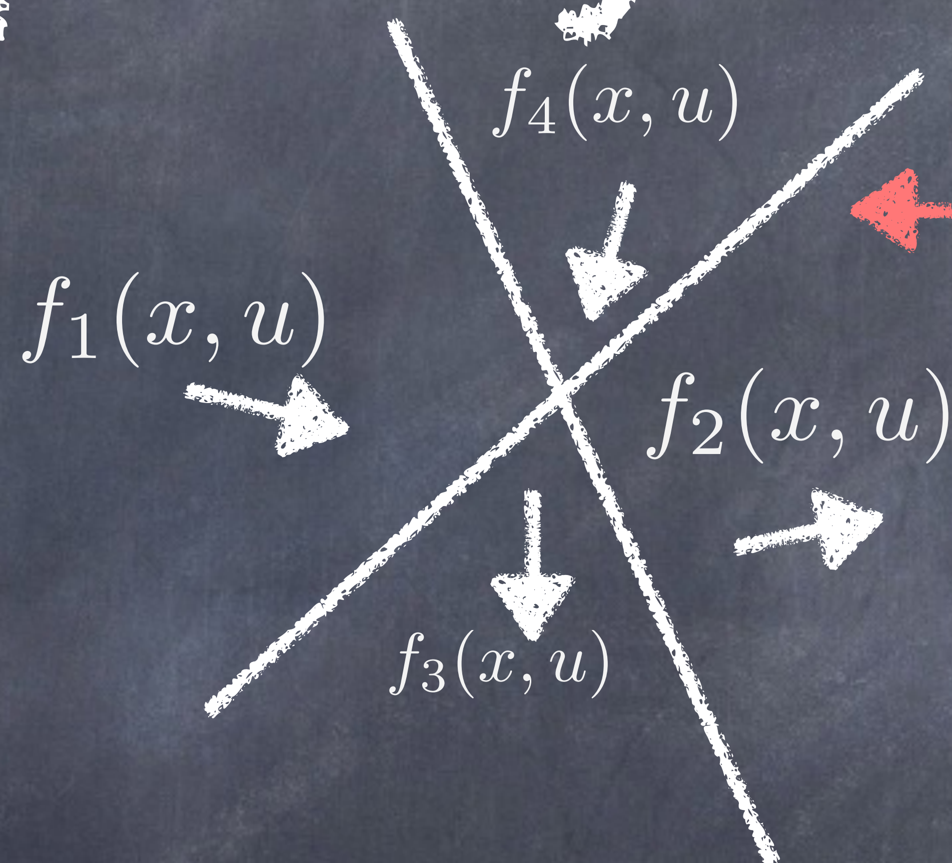


From A. Ames, multiple papers

Formalism 2: Filippov system

state → x, input → u

$$\dot{x} = f_q(x, u)$$

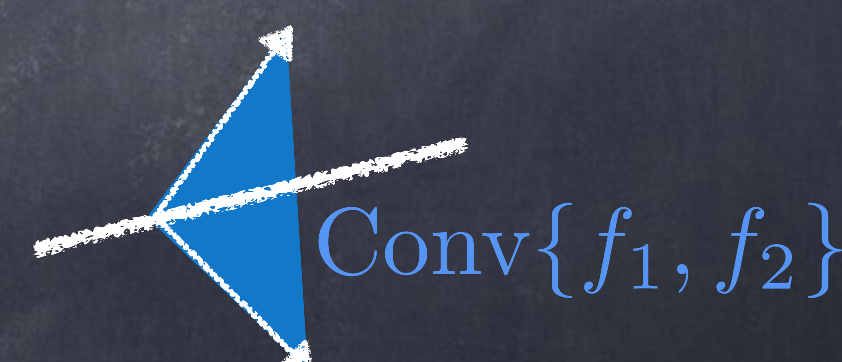


Assumption: the guards PARTITION the state space X

$$\text{if } x \in G_q \text{ then}$$

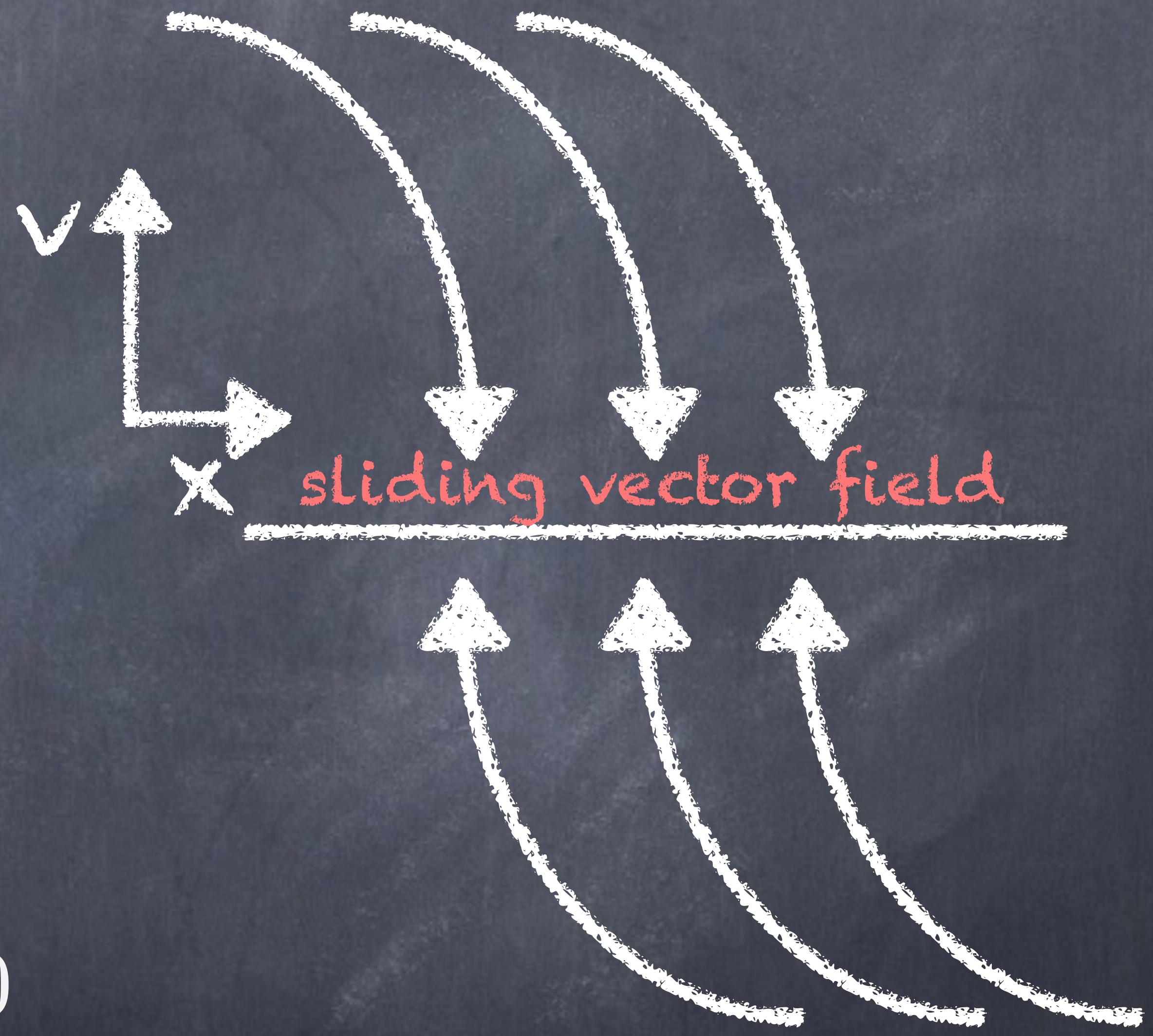
G_q ← discontinuity set

$$f(x,u) \in \text{Conv}\{f_{q_1}(x,u), f_{q_2}(x,u), \dots\}$$



the state of the system is x

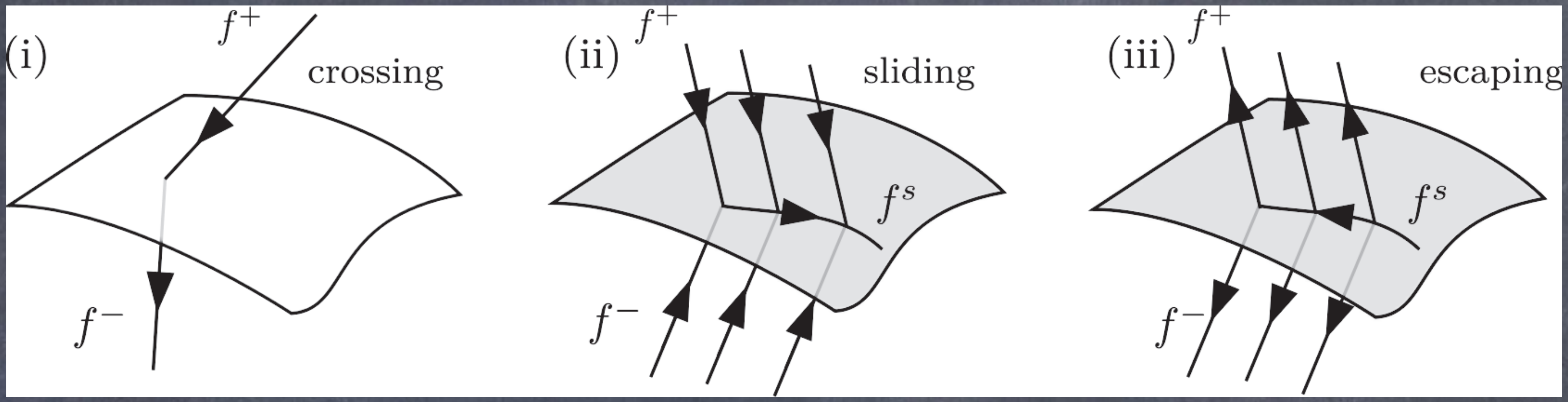
Example: sliding mass as a Filippov system

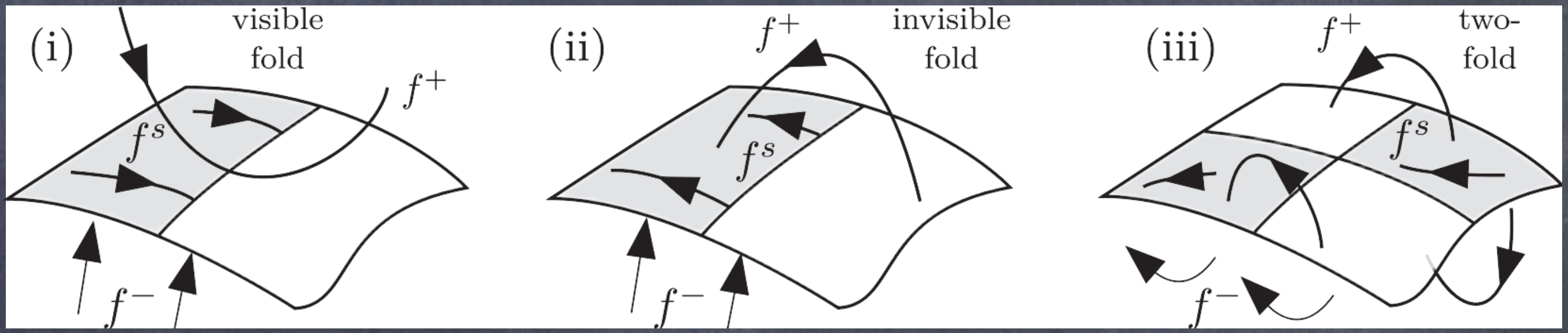


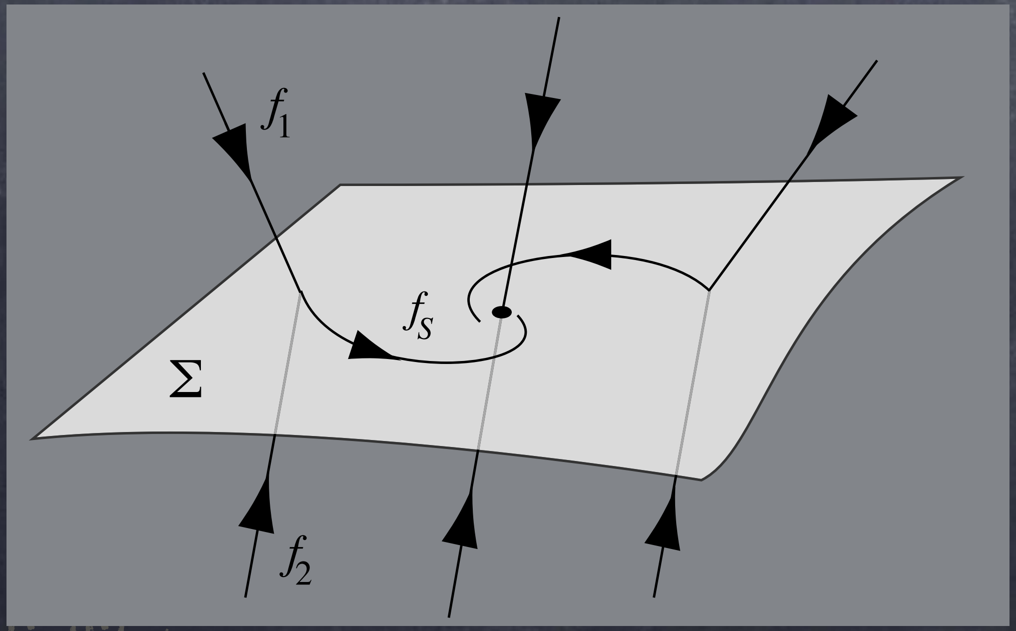
$$\dot{x} = v$$

$$\dot{v} = \begin{cases} \frac{f_g}{m}\sin(\theta) - C \text{ if } v > 0 \\ \frac{f_g}{m}\sin(\theta) + C \text{ if } v < 0 \\ \frac{f_g}{m}\sin(\theta) + [-C, C] \text{ if } v = 0 \end{cases}$$

Sliding, crossing, escaping, equilibria, and pseudoequilibria

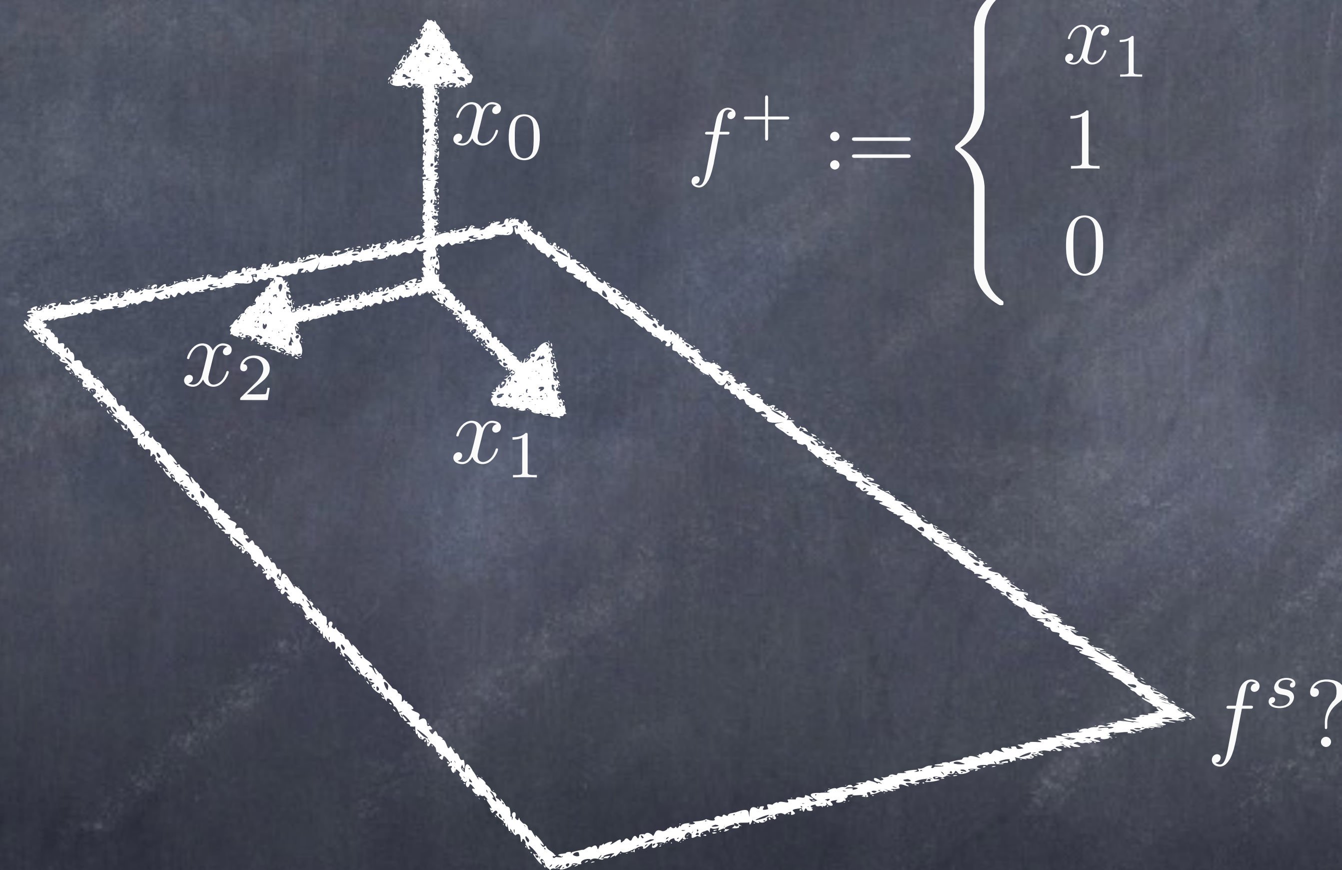






pseudoequilibrium

Example: compute the sliding vector field of a Filippov system formed by 2 linear subsystems



$$f^+ := \begin{cases} x_1 \\ 1 \\ 0 \end{cases}$$

$$f^- := \begin{cases} 1 \\ 0 \\ 0 \end{cases}$$

Example: compute the sliding vector field of a Filippov system formed by 2 linear subsystems

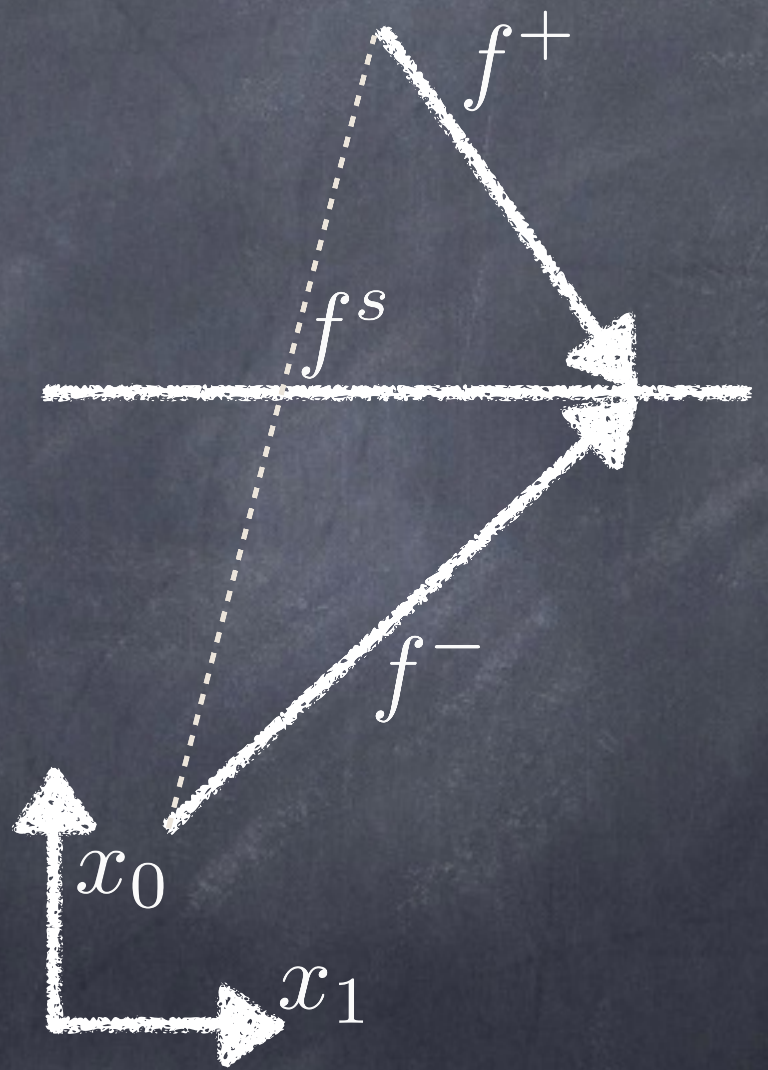
$$f^s = \lambda f^+ + (1-\lambda)f^-$$

λ such that $\lambda f^+ + (1-\lambda)f^-$ is tangent to the discontinuity surface

$$\lambda f_0^+ + (1-\lambda)f_0^- = 0$$

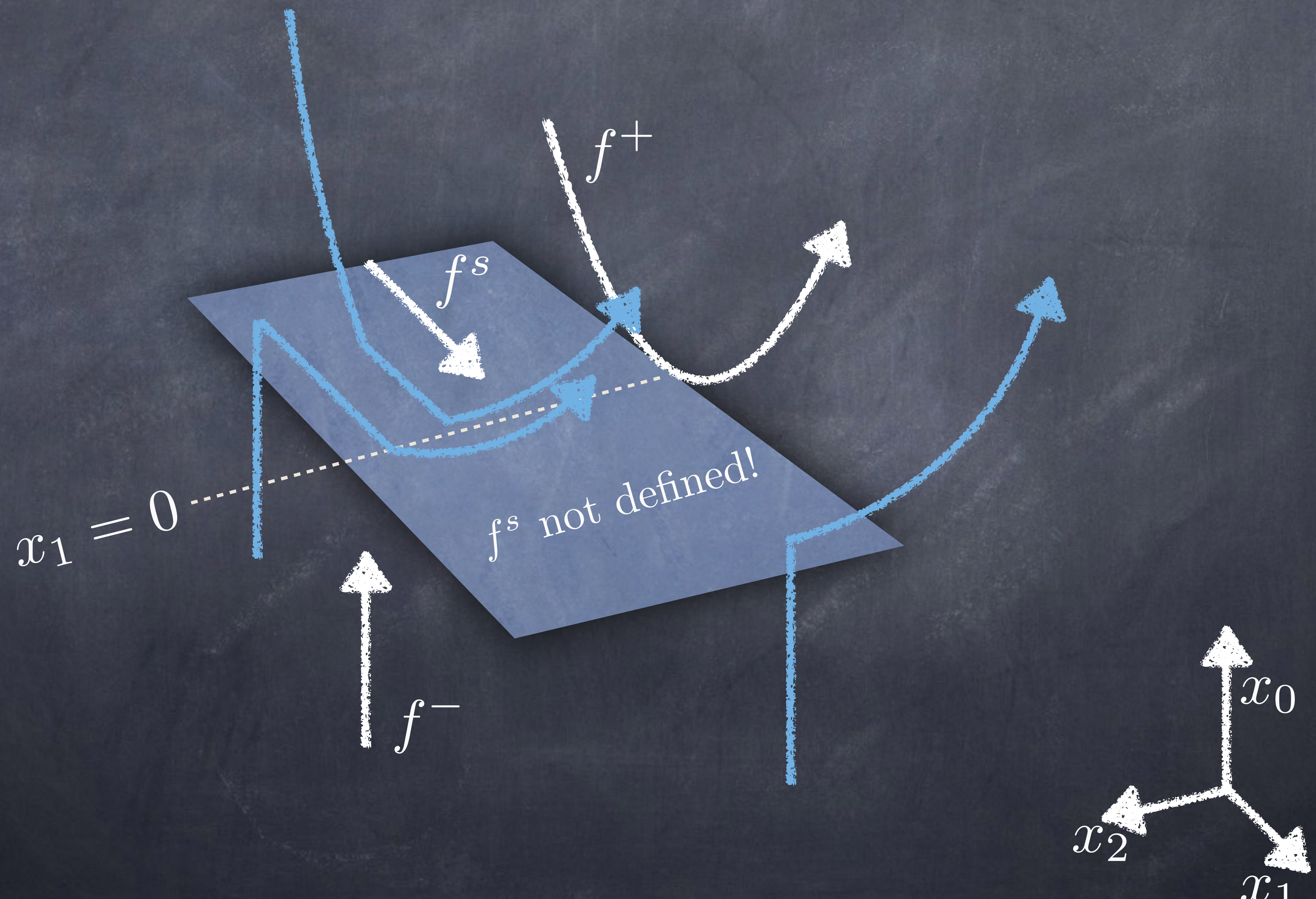
$$\lambda x_1 + (1-\lambda)1 = 0$$

$$\lambda = \frac{1}{1-x_1}$$

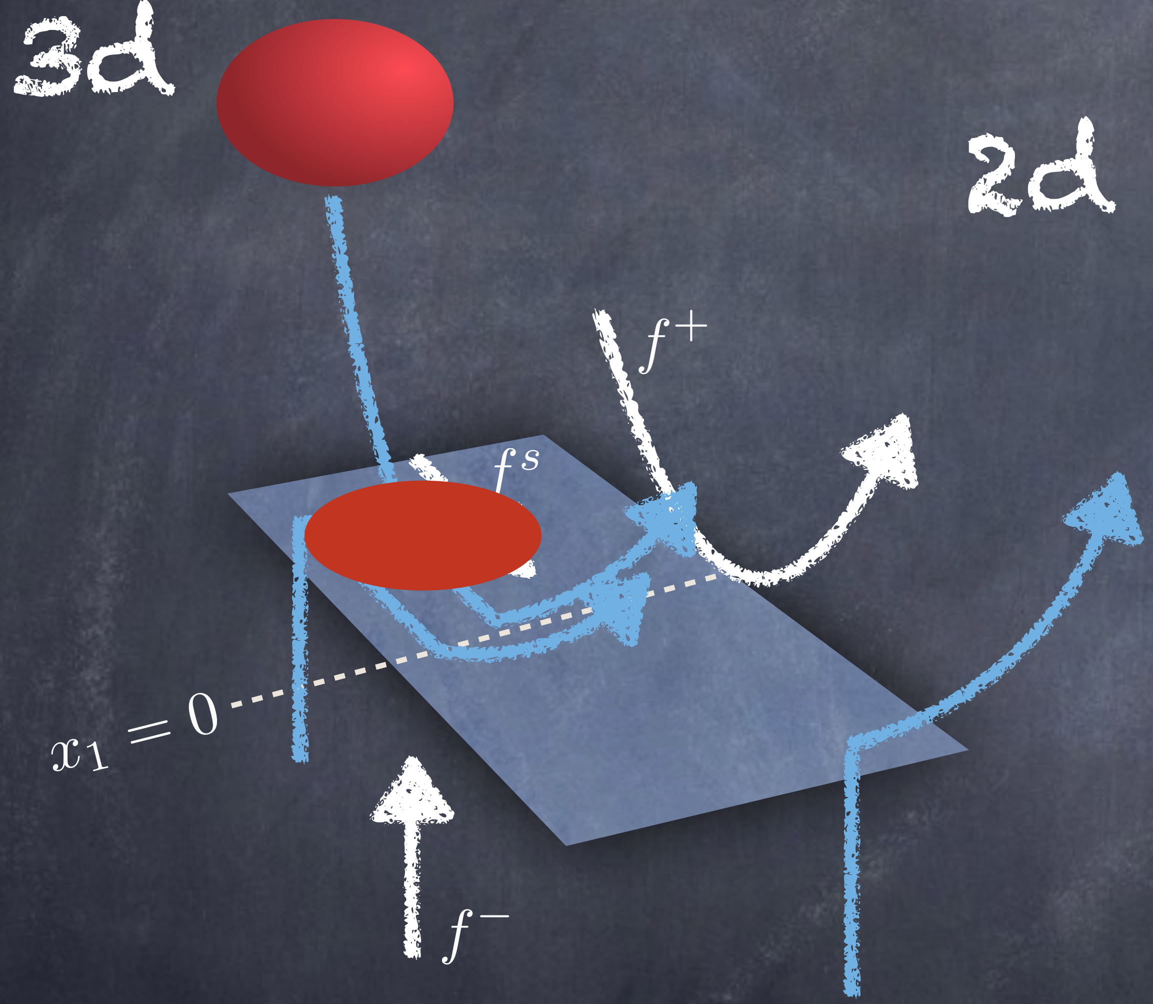


$$f^- := \begin{cases} 1 \\ 0 \\ 0 \end{cases} \qquad f^+ := \begin{cases} x_1 \\ 1 \\ 0 \end{cases}$$

$$f^s = \frac{1}{1-x_1}\begin{pmatrix} x_1 \\ 1 \\ 0 \end{pmatrix} + \frac{x_1}{x_1 - 1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{1-x_1} \\ 0 \end{pmatrix}$$



Thought experiment



What happens to the Lyapunov exponents of a sliding trajectory?

Other formalisms

- Complementarity system
- Differential inclusion
- Switched system

Common friction models

Coulomb model

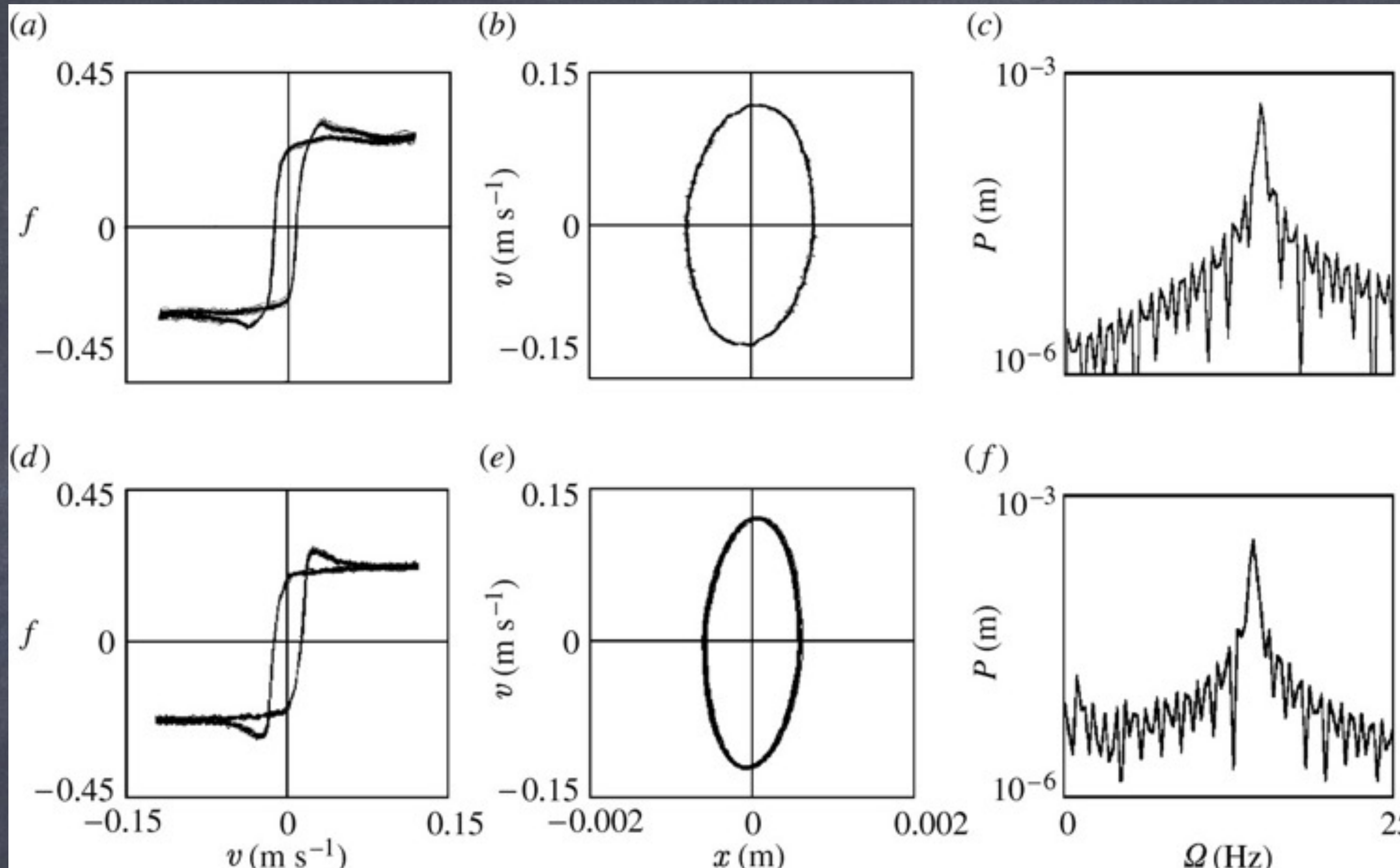
Stiction (static friction)

Stribeck effect

hysteresis

experimental

numerical

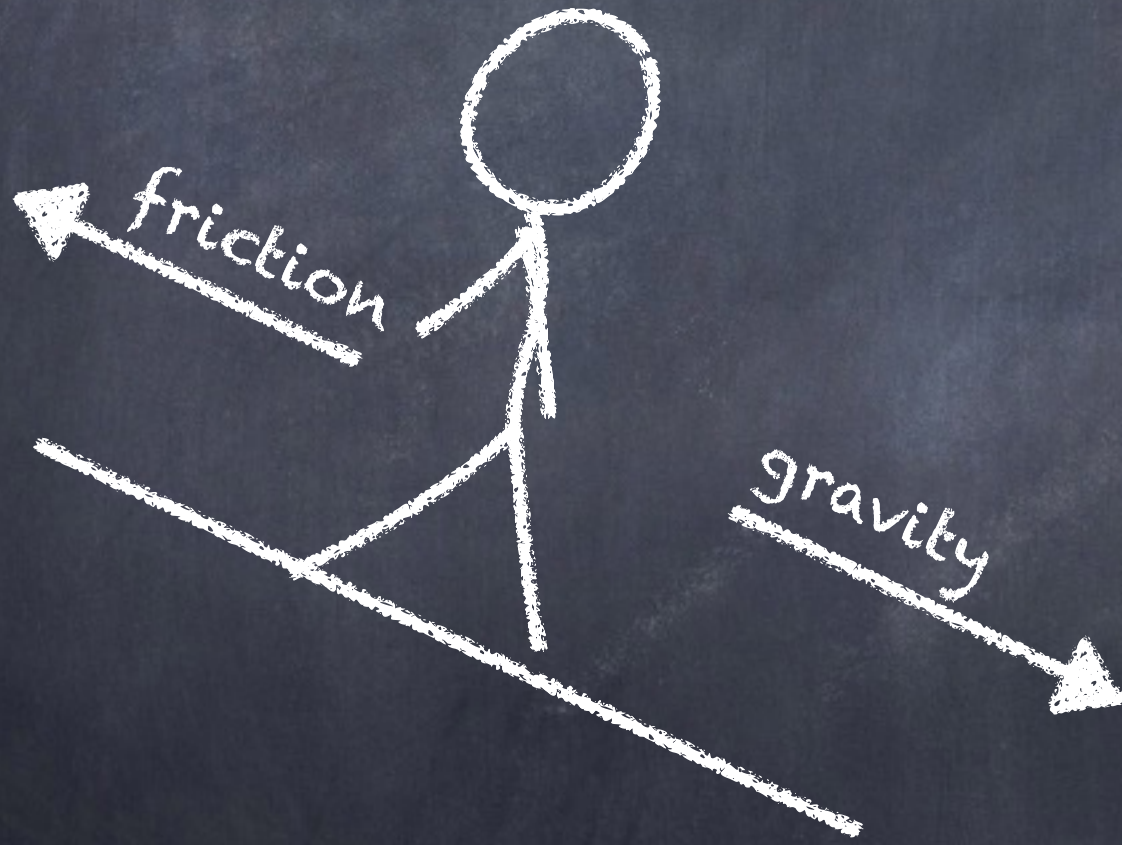


$$f(v,\dot{v}) = \begin{cases} f_{st}\,\mathrm{sgn}(v) \text{ for } f_{st} < f_{d+} \text{ and } \mathrm{sgn}(v\dot{v}) > 0, \\ f_{d+}\,\mathrm{sgn}(v) \text{ for } f_{st} > f_{d+} \text{ and } \mathrm{sgn}(v\dot{v}) > 0, \\ f_{d-}\,\mathrm{sgn}(v) \text{ for } \mathrm{sgn}(v\dot{v}) < 0, \end{cases}$$

(from "Hysteretic effects of dry friction: modelling and experimental studies"
J, Wojewoda, A. Stefański, M. Wiercigroch and T. Kapitaniak, 2008)

Case study: phase portrait of a man standing on an increasingly steep surface

(with a Filippov model and Stribeck friction)



Stribeck friction characteristic

Case study: phase portrait of a man walking on an increasingly steep surface

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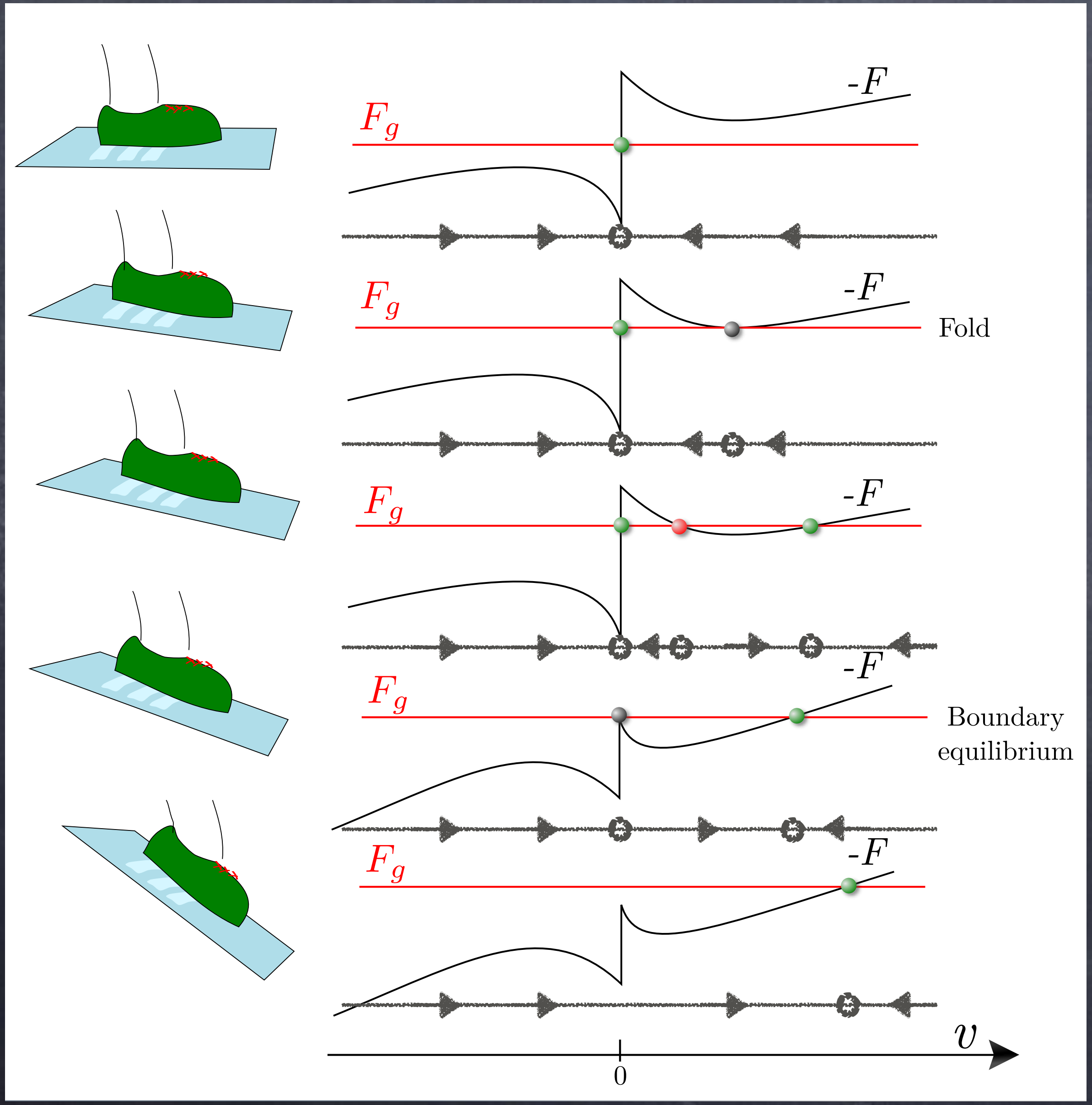

A new, non smooth bifurcation: the Boundary Equilibrium (more to come...)