POLITECNICO DI MILANO

# Bifurcation analysis of the attitude dynamics for a magnetically controlled spacecraft


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# Outline



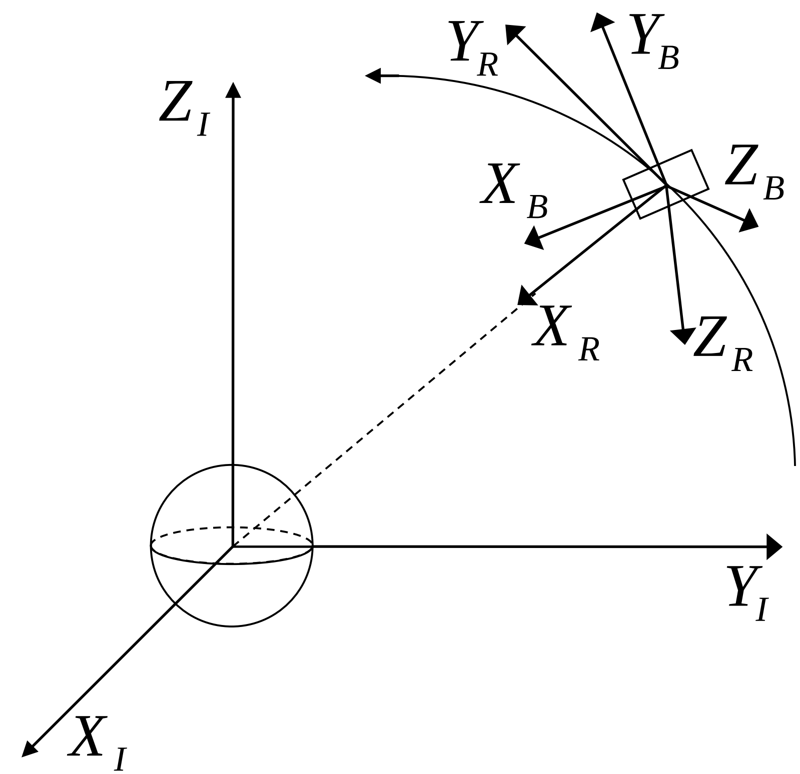
- Spacecraft models and magnetic control
- Local stability analysis
- Bifurcation analysis
- Simulation results
- Concluding remarks and future works

# Introduction to magnetic attitude control

- Attitude control plays a fundamental role in the operation of a satellite.

- For conventional actuators (e.g., reaction wheels, thrusters) considerable work has been done for both the local and the global control problems.

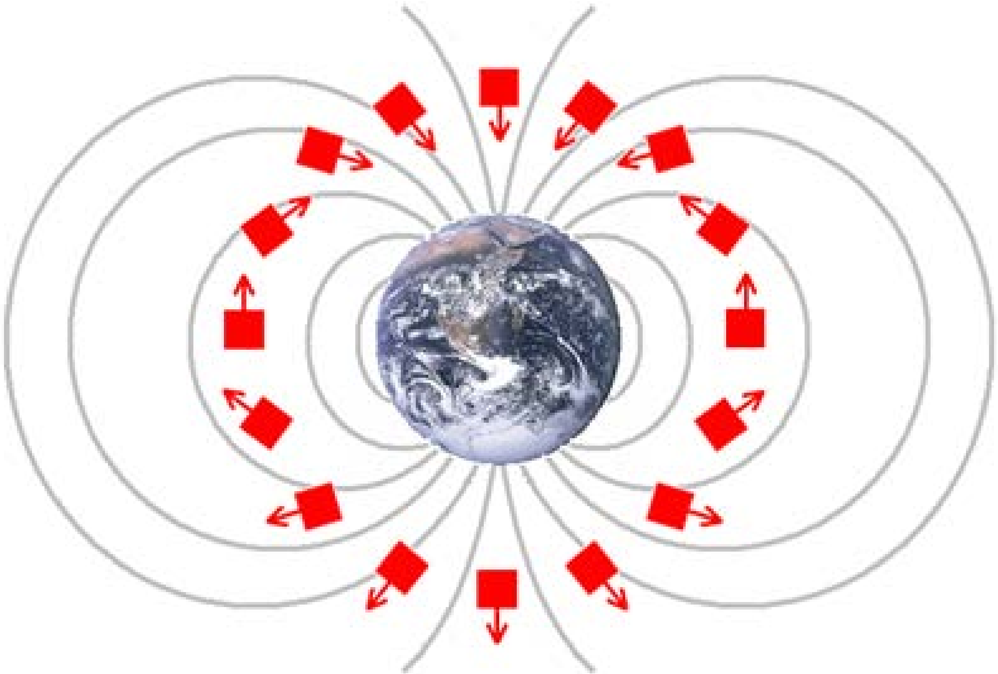
- On the other hand, while magnetic coils have been extensively used in practice, limited attention has been dedicated to the underlying theoretical issues.



## Critical issues in magnetic control

Magnetic actuators are intrinsically time-varying

It is not possible to provide three independent control torques at each time instant



Attitude stabilisation is possible because **on average** the system possesses strong controllability properties for a wide range of orbit inclinations

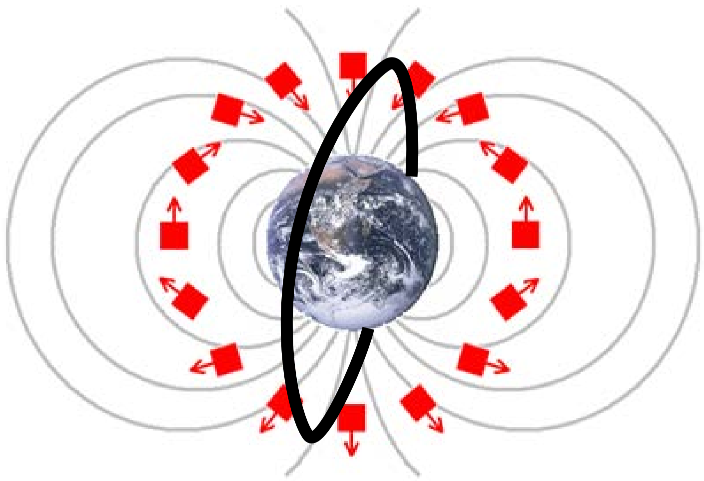
THANKS TO: UNIVERSITY OF MICHIGAN – CUBESAT PROGRAM

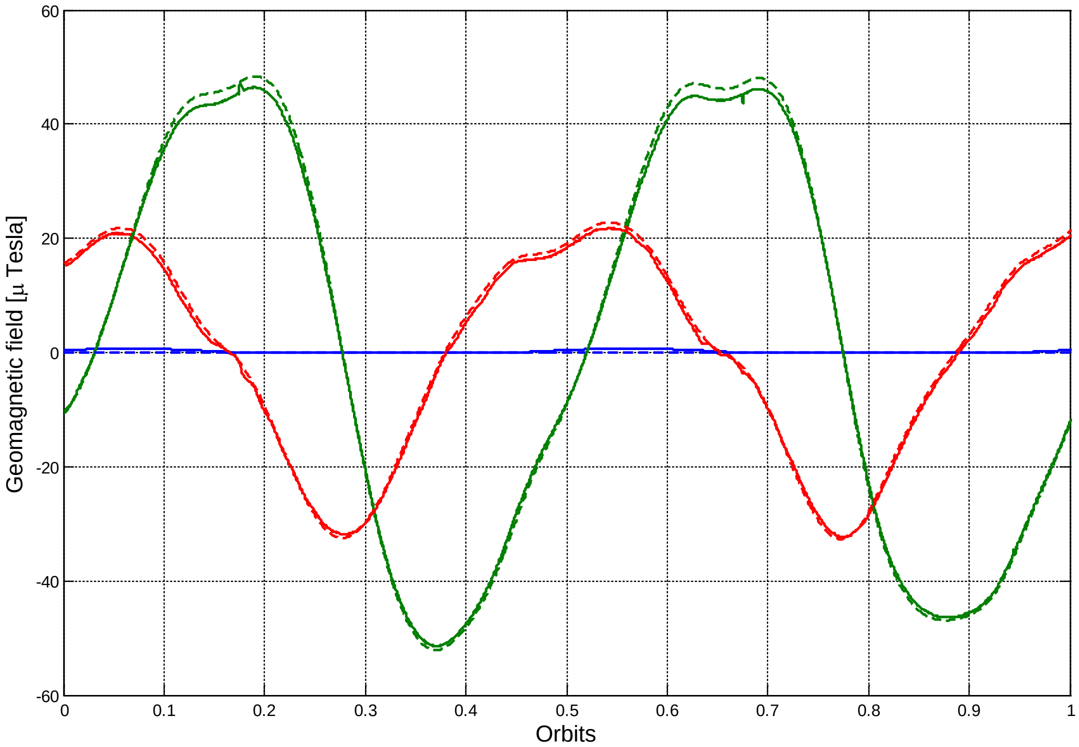
# Geomagnetic field

The Earth's magnetic field can be easily

- modelled: see, e.g., the International Geomagnetic Reference Field model;
- measured on board;

- **MITA spacecraft**
- **Near polar (87° inclination);**
- **Altitude of 450 km;**





# Magnetic torques

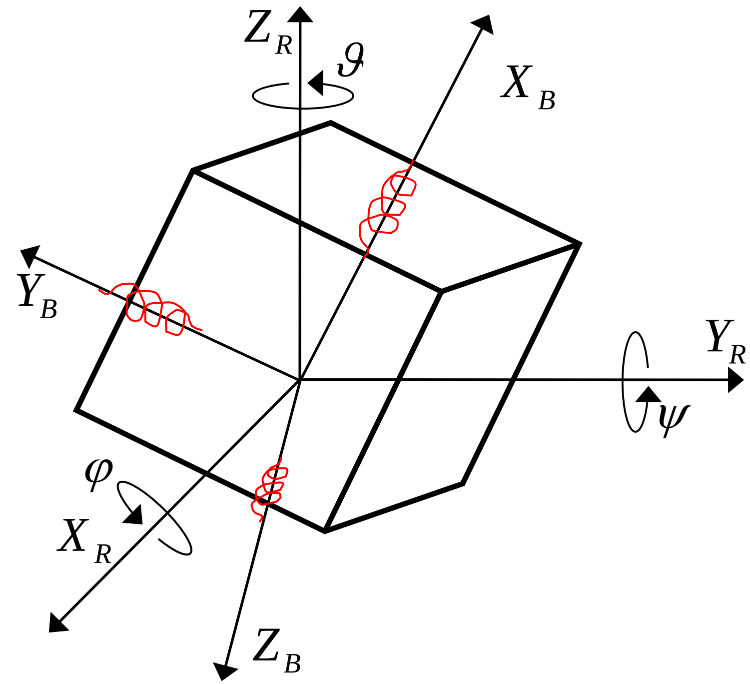


The magnetic attitude control torques $T_{coils}$ are given by:

$$T_{coils} = m_{coils} \times b(t) = S(b(t))m_{coils}$$

where:

- $$S(\omega) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}$$
- $m_{coils}$ $2\ \mathrm{R}^3$ magnetic dipoles for the three current-driven coils




- *b(t)* $2\ \mathrm{R}^3$ Earth magnetic field versor (body frame), given by

$$b(t) = A(\mathbf{q})b_0(t)$$

- ***q*** is an attitude parametrization (quaternion) $q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T = \begin{bmatrix} q_r^T & q_4 \end{bmatrix}^T$
- *A(**q**)* is the attitude matrix $A(\mathbf{q}) = (q_4^2 - |q_r|^2)\mathbb{I}_3 + 2q_r q_r^T + 2q_4 S(q_r)$
- $b_0(t)$ is the magnetic field vector in orbit coordinates

# Magnetic attitude control: discussion



Advantages of magnetic control:

- low cost actuators;
- no moving parts (e.g., reaction wheels);
- low power consumption, no fuel (e.g., thrusters).

Limitations:

- viable only for Low Earth Orbit spacecraft (<1000 km altitude);
- it is not possible to full control the spacecraft at each time instant;
- controller analysis and design is more complicated.

# Model for rigid body angular motion (body frame)

The overall dynamics is given by the kinematics of the spacecraft

$$\dot{\mathbf{q}} = W(\mathbf{q})\omega \qquad W(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

and by the attitude dynamics

$$I\dot{\omega} = S(\omega)I\omega + T_{coils} = S(\omega)I\omega + S(A(\mathbf{q})b_0(t))m_{coils}$$

## Problems:

- show that the system is controllable **on average**;
- work out a globally stabilizing control law.

# State feedback regulation

For the *projection-based* control law

$$m_{coils} = -S^T(b(t)) I^{-1} (\varepsilon^2 k_p \mathbf{q} + \varepsilon k_v \omega)$$

using averaging techniques it is possible to obtain both results, i.e.,

- the system is controllable on average if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(b_0(t)) S^T(b_0(t)) dt > 0$$

- there exist $\varepsilon^F > 0$, $k_p > 0$, $k_v > 0$, such that for any $0 < \varepsilon < \varepsilon^F$ the control law renders the equilibrium $(q,\omega) = ([0\ 0\ 0\ 1]^T, 0)$ of the closed loop system locally exponentially stable.

- moreover, all trajectories of the closed loop system converge to the points $(q,\omega) = (\S\ [0\ 0\ 0\ 1]^T, 0)$.

# Sketch of proof

Introduce the coordinate transformation

$$z_1 = q \quad z_2 = \frac{\omega}{\varepsilon}$$

Set $\Gamma(t) = \frac{1}{\|b\|^2} S(b) S^T(b) = \frac{1}{\|b\|^2}(I - bb^T) \geq 0$

and write the equations (in the inertial body frame) of motion as

$$\begin{aligned} \dot{z}_1 &= \varepsilon \tilde{W}(z_1) z_2 \\ \dot{\widehat{Iz_2}} &= \varepsilon \Gamma_0(t) I^{-1}(-k_p z_{r1} - k_v z_2) \end{aligned}$$

consider the averaged system

$$\begin{aligned} \dot{z}_1 &= \varepsilon \tilde{W}(z_1) z_2 \\ \dot{\widehat{Iz_2}} &= \varepsilon \Gamma_{0av} I^{-1}(-k_p z_{r1} - k_v z_2) \end{aligned}$$

and use the Lyapunov function

$$V_1(z_1, z_2) = \frac{1}{2} k_p (z_{r1}^T z_{r1} + (z_{14} - 1)^2) + \frac{1}{2} (I z_2)^T \Gamma_{0av}^{-1} (I z_2).$$

# State feedback regulation

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- moreover, all trajectories of the closed loop system converge to the points $(q,\omega) = (\S\ [0\ 0\ 0\ 1]^T,0)$.

**But which is a suitable choice for $\varepsilon$?**

# Local stability analysis

First approach: analyze the local stability of the

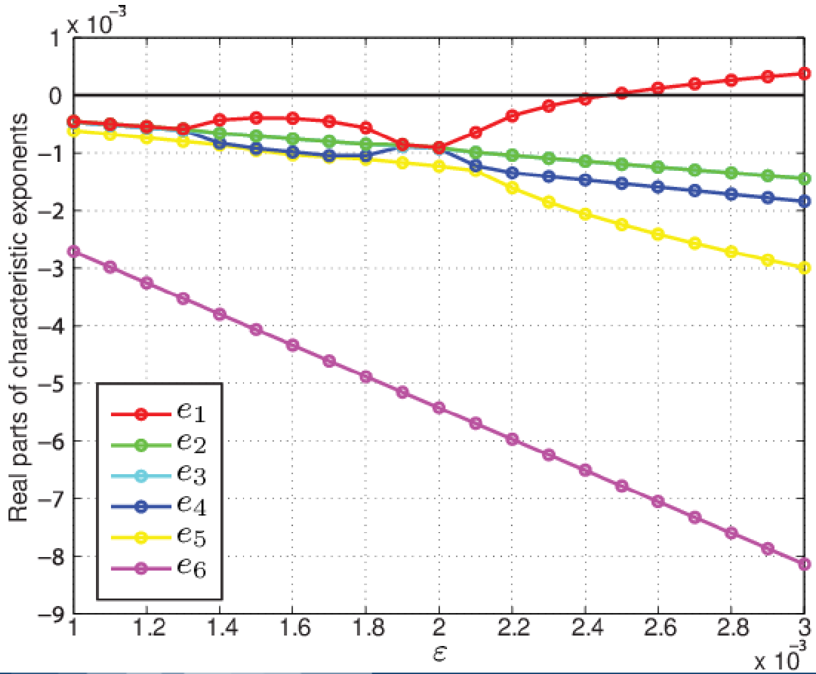
$(q,\omega)=([0\ 0\ 0\ 1]^T,0)$

equilibrium of the closed loop periodically forced system.

Step 1: Linearize the closed-loop system around the equilibrium

$$\begin{aligned} I\dot{\omega} &= -S(b_0(t))S^T(b_0(t))I^{-1}(\varepsilon^2 k_p \mathbf{q} + \varepsilon k_v \omega) \\ \dot{q}_r &= \frac{1}{2}\omega \\ \dot{q}_4 &= 0 \end{aligned}$$

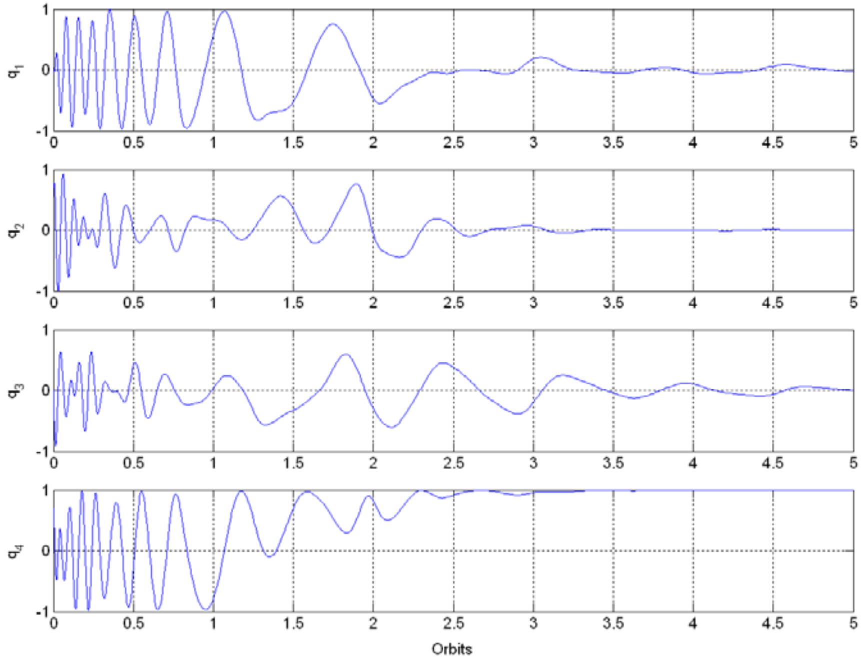
Step 2: Study the stability of this linear time-periodic system through Floquet theory

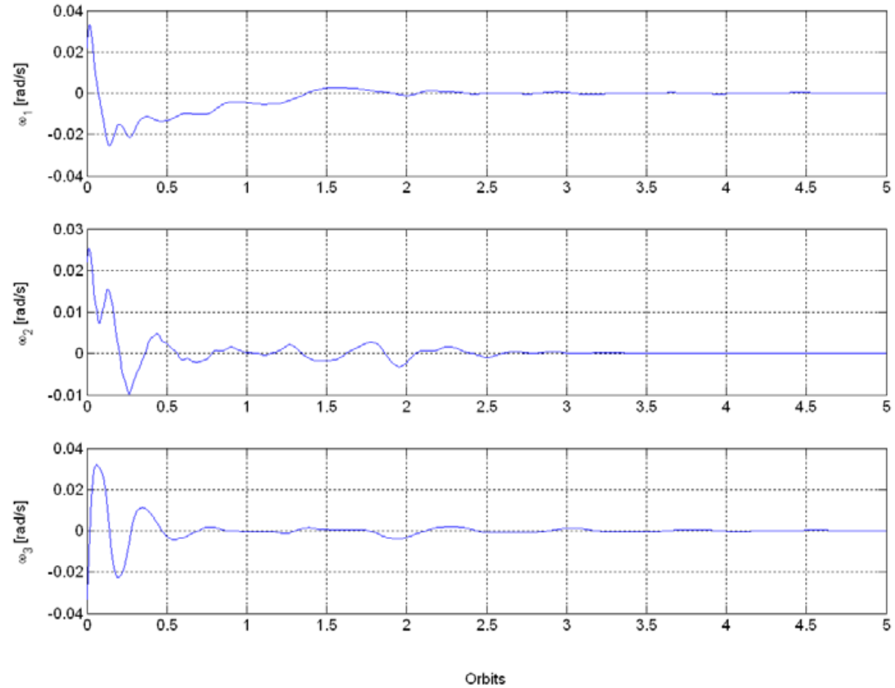


Considered satellite:

$$\varepsilon = 0.001$$

- Inertia matrix I=diag[27, 17, 25] Nm;
- Near polar (87º inclination) orbit with altitude of 450 km and orbit period of about 5600 s.



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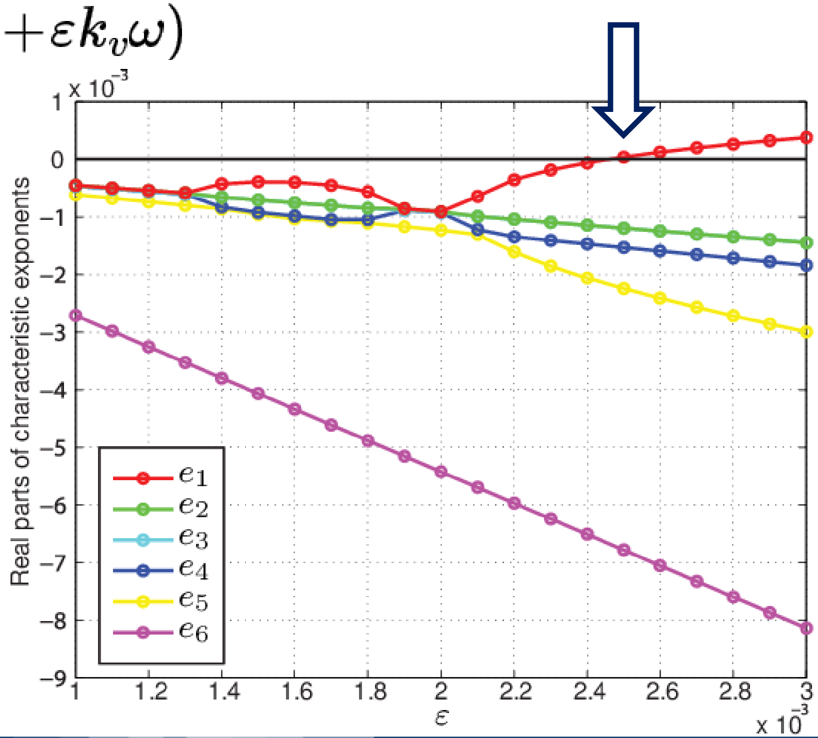


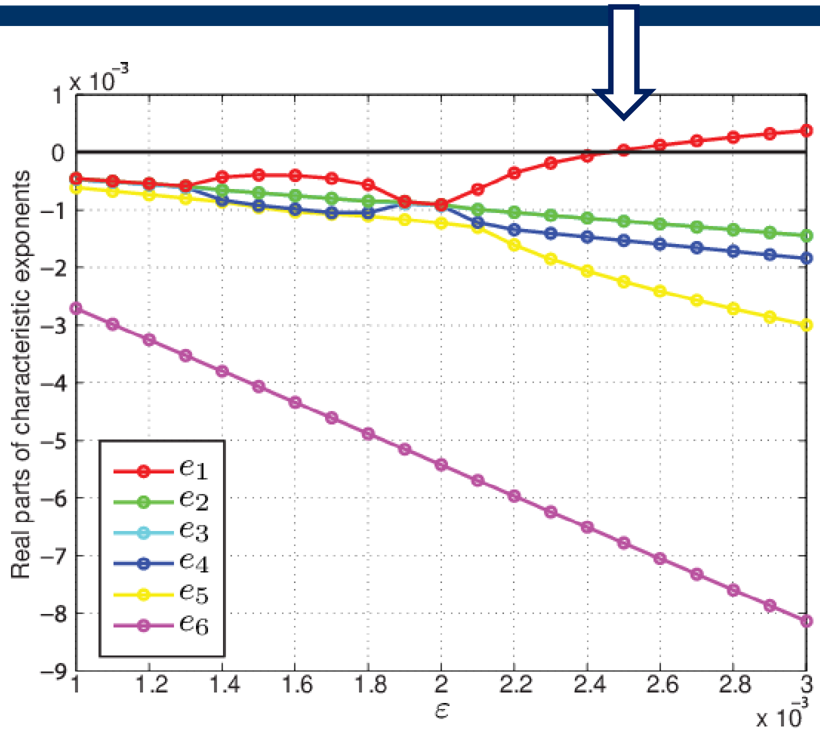

Step 2: Study the stability of this linear time-periodic system through Floquet theory

**What happens when local stability is lost?**

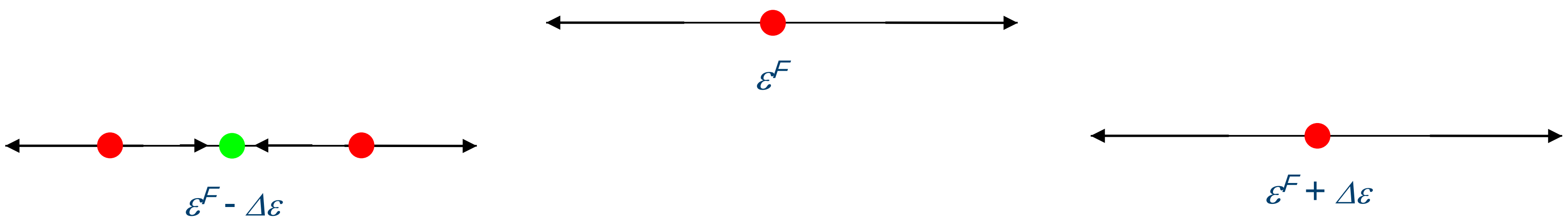
# CHANGE IN LOCAL STABILITY → (LOCAL) BIFURCATION

In our case we have only an eigenvalue that becomes unstable.

This means that we can analyze what happens in a neighborhood of the bifurcation by studying a one dimensional system (center manifold theorem)




Nonlinear term analysis shows that at the bifurcation the equilibrium is unstable




Thus, near the bifurcation the stationary solution is not globally stable

System equations read:

$$\dot{\mathbf{q}} = W(\mathbf{q})\omega \qquad W(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

$$I\dot{\omega} = S(\omega)I\omega + T_{coils} = S(\omega)I\omega + S(A(\mathbf{q})b_0(t))m_{coils}$$

$$m_{coils} = -S^T(b(t))I^{-1}(\varepsilon^2 k_p \mathbf{q} + \varepsilon k_v \omega)$$

Problems for numerical continuation:

- The system is time-dependent (periodically forced)
- The system live in SO3 (||**q**|| = 1)

# Setting up the system for numerical continuation

$$I\dot{\omega} = S(\omega)I\omega + T_{coils} = S(\omega)I\omega + S(A(\mathbf{q})b_0(t))m_{coils}$$

$$m_{coils} = -S^T(b(t))I^{-1}(\varepsilon^2 k_p \mathbf{q} + \varepsilon k_v \omega)$$

Problems for numerical continuation:

- The system is time-dependent (periodically forced)

$$b_0(t) = \begin{bmatrix} -1.8806 & -6.2168 & -36.0046 \\ -1.0787 & -0.4385 & -1.0345 \\ -11.7084 & 35.6763 & -6.4530 \end{bmatrix} \begin{bmatrix} 1 \\ \cos\omega t \\ \sin\omega t \end{bmatrix}$$

We can therefore substitute the periodic dependence introducing two variables that are solution of the sistem

$$\dot{cos} = cos - \omega sin - (cos^2 + sin^2)cos$$

$$\dot{sin} = \omega cos + sin - (cos^2 + sin^2)sin$$

# Setting up the system for numerical continuation

$$\dot{\mathbf{q}} = W(\mathbf{q})\omega \qquad W(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

Problems for numerical continuation:

- The system live in SO3 (||**q**|| = 1)

System equations are written in order to maintain $\mathbf{q}^2 = 1$, since

$$\frac{d}{dt} q^2(t) = 2q\dot{q} = qW(q)\omega = 0\omega$$

but this manifold is not attractive (there is an eigenvalue equal to 0!).

We can so:

- Exploit the algebraic constrain in order to eliminate a variable

$$q_4 = \pm\sqrt{1 - q_1^2 - q_2^2 - q_3^2}$$

- Make the manifold $\mathbf{q}^2 = 1$ stable introducing a dumping

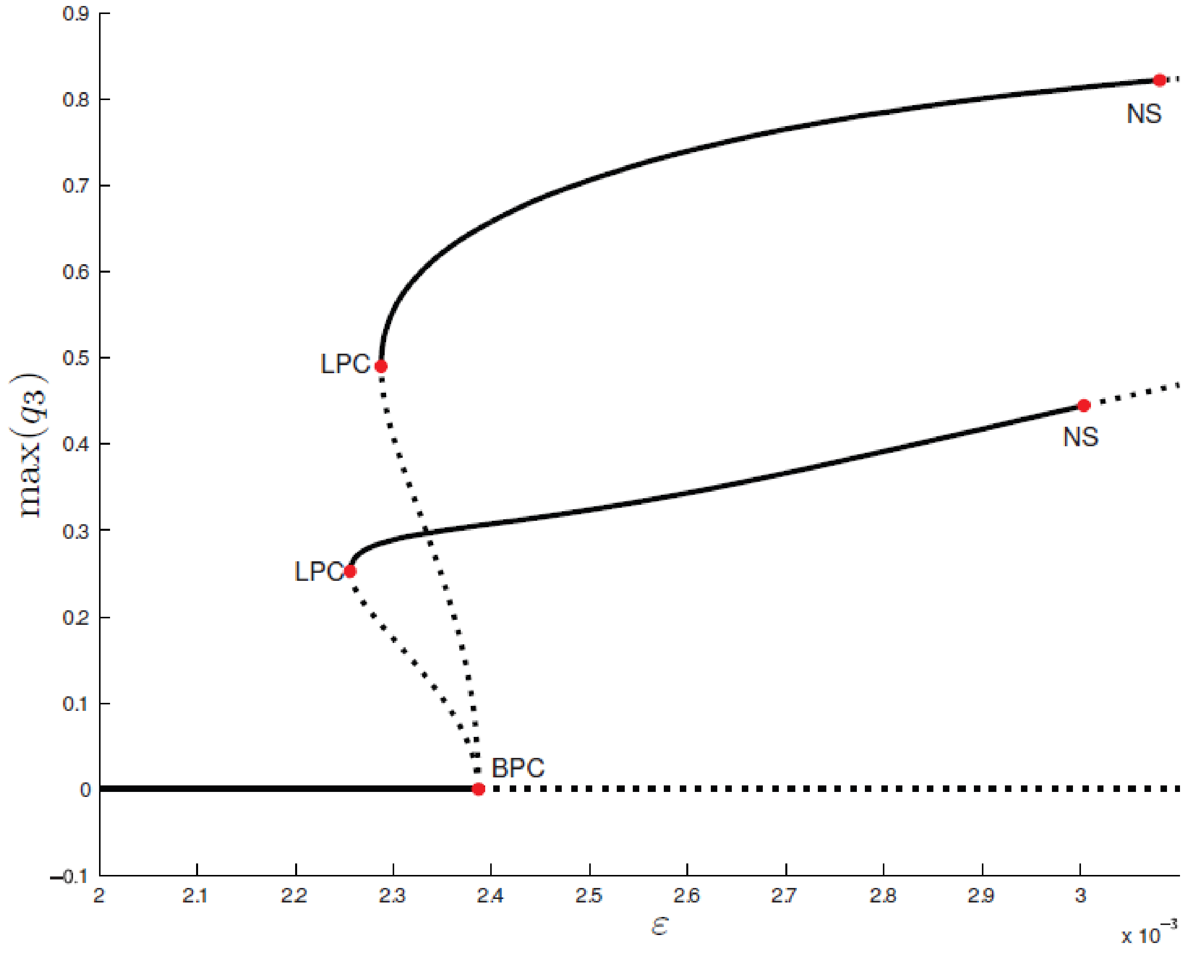
$$\dot{q} = W(q)\omega + q(1 - \|q\|^2)$$

# Try it with MatCont

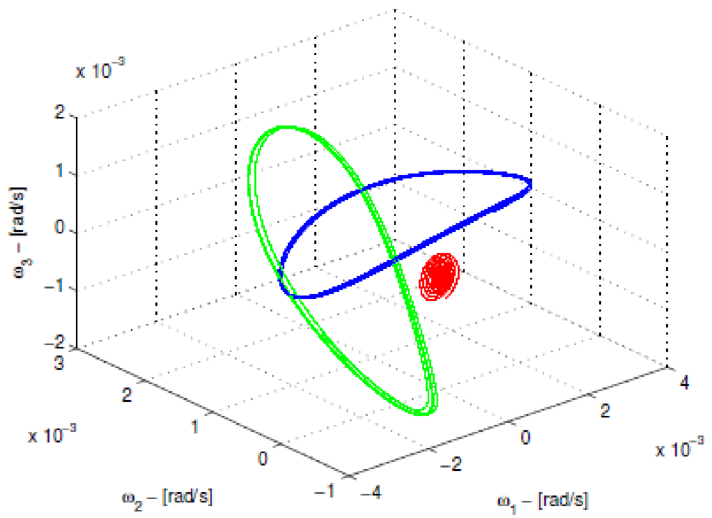
# Numerical analysis results

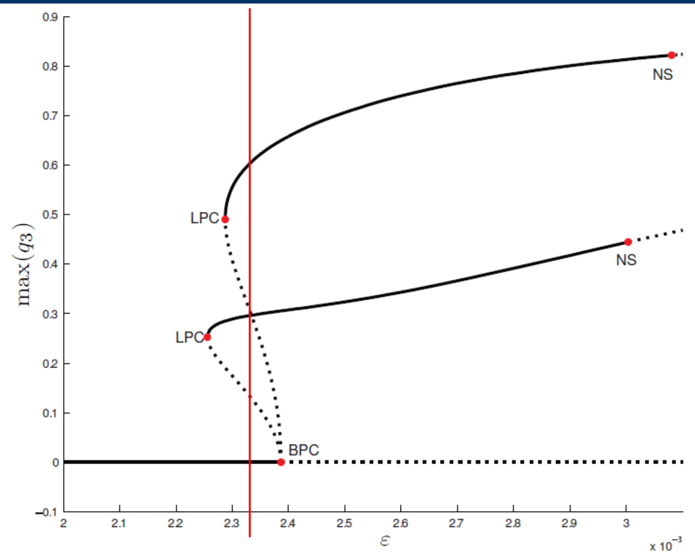
Using continuation techniques it is possible to analyze the solutions born at the bifurcation, obtaining a diagram of the invariants for different values of the parameter.

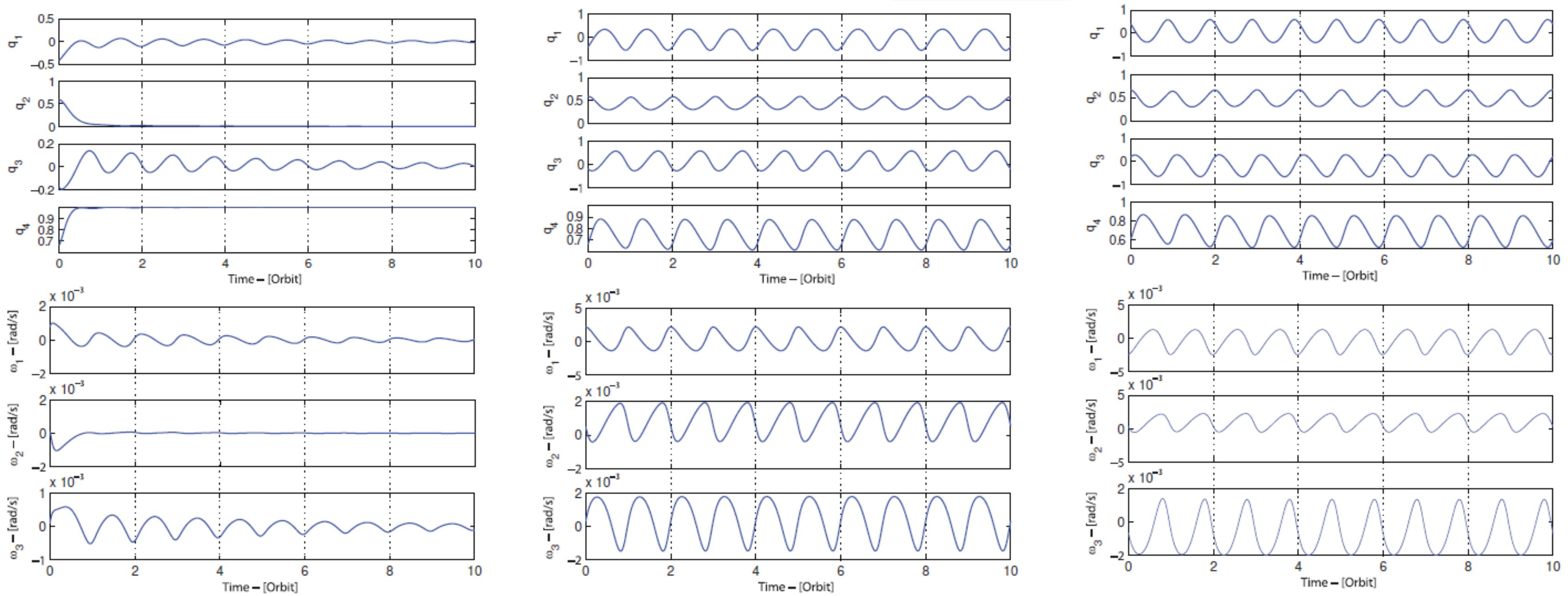
Note: since the satellite moves along the orbit all the invariants (even the stationary one) are limit cycles.



# Numerical analysis results

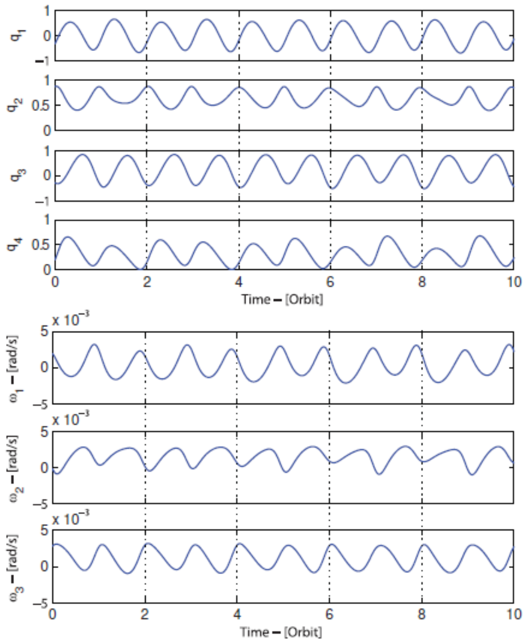


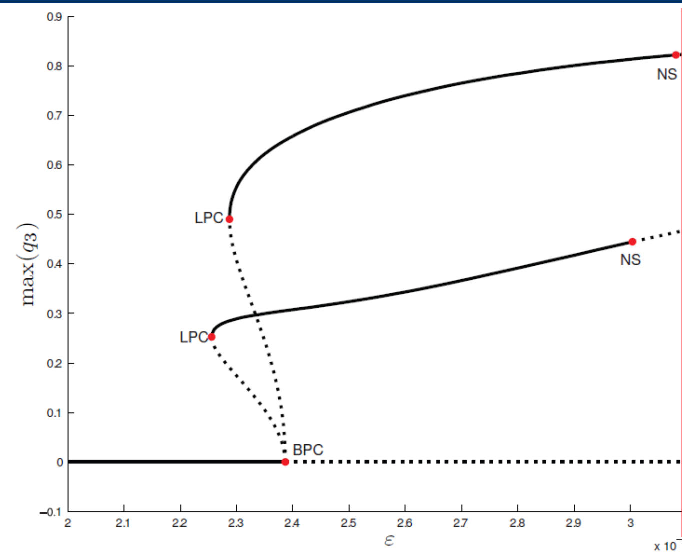



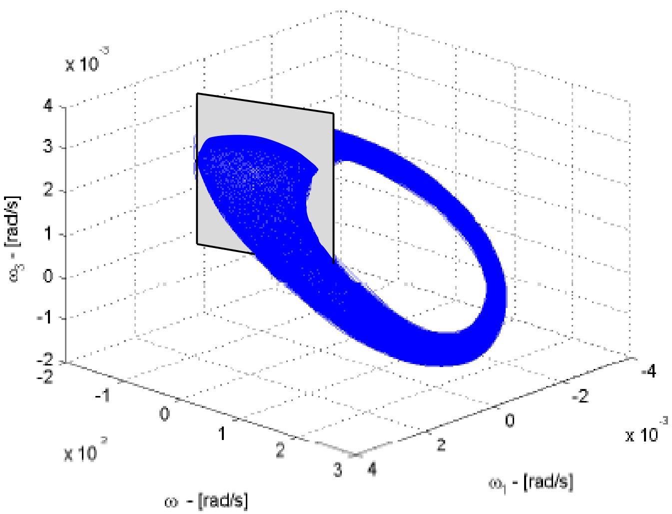



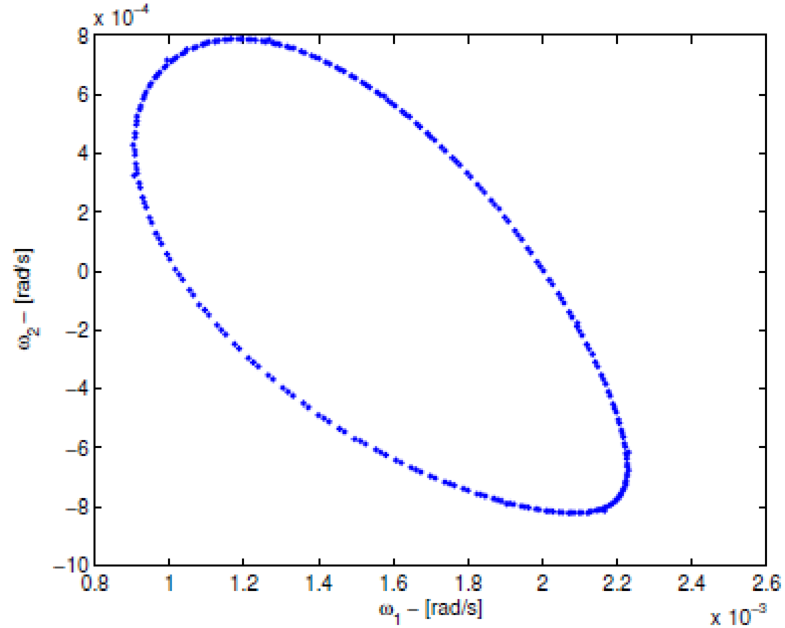
# Numerical analysis results

$L_1$ = 0.00001456
$L_2$ = - 0.0000743
$L_3$ = - 0.0642956
…



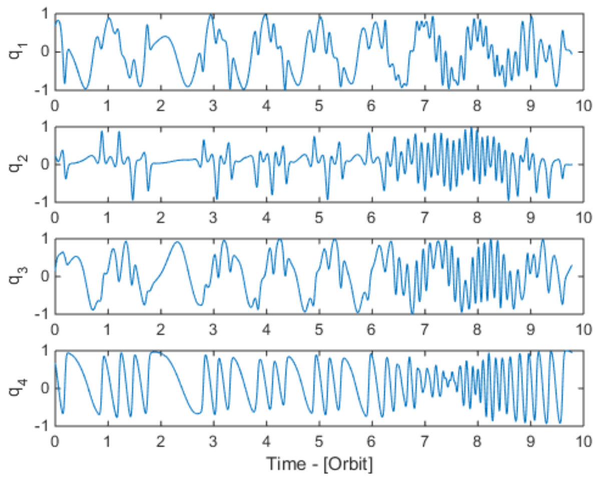



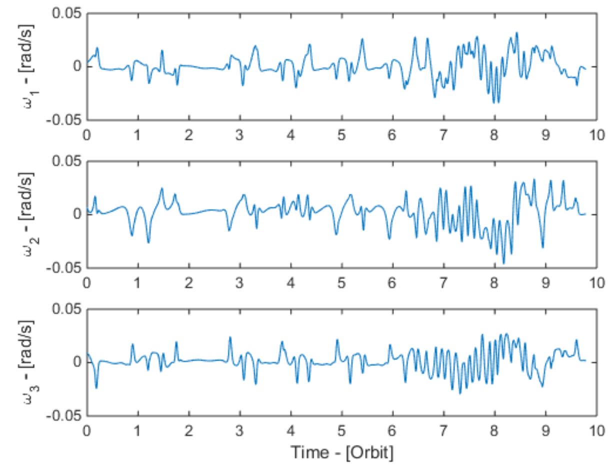



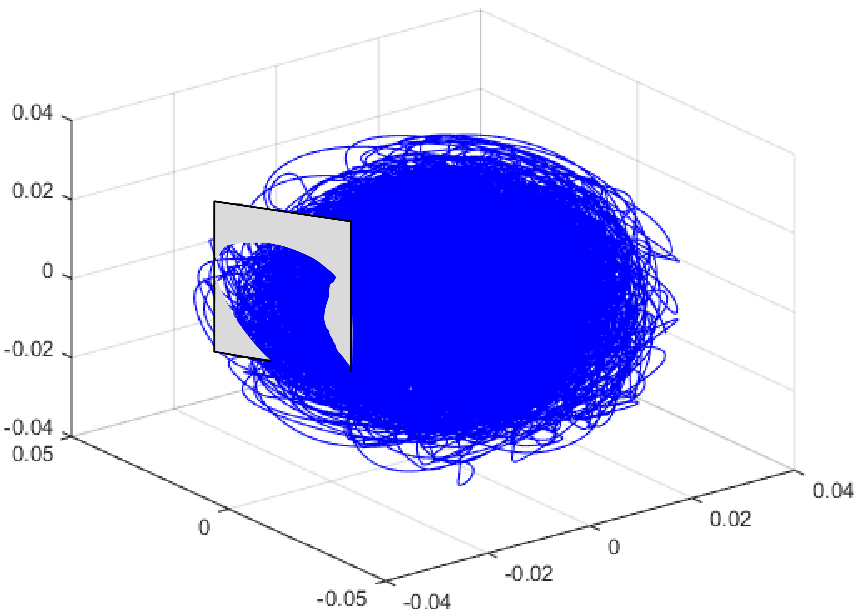
# Numerical analysis results

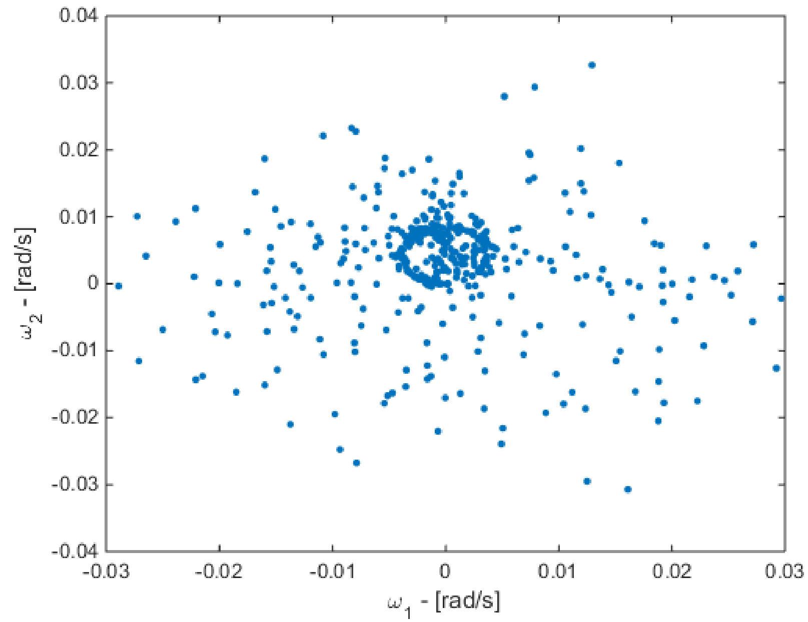
$L_1 = 0.0245456$
$L_2 = -\ 0.0000521$
$L_3 = -\ 0.0349725$
…




$\varepsilon = 0.03$





# Conclusions



- A control law for magnetic attitude regulation developed in previous work has been considered
- The control law provides global convergence to the desired equilibrium if the control strength is small enough
- Local stability of the desired equilibrium has been analysed and an upper bound to the control strength has been worked out
- Nonlinear analysis has been applied to better understand the closed-loop system, finding:
  - multistability
  - a new (more restrictive) bound to ensure global stability
  - quasi-periodic and chaotic behaviors if the control strength is too large.