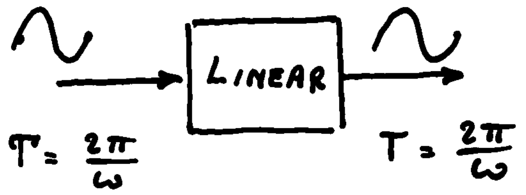
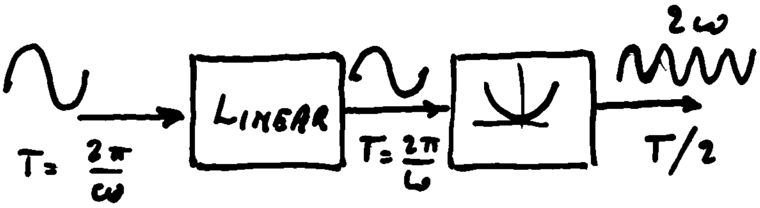
Harmonics and subharmonics



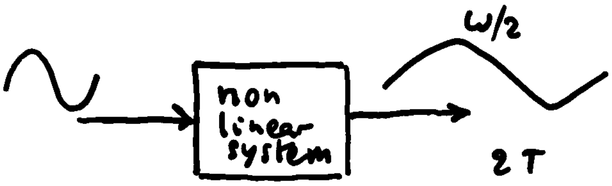
input and output frequencies are the same



harmonics

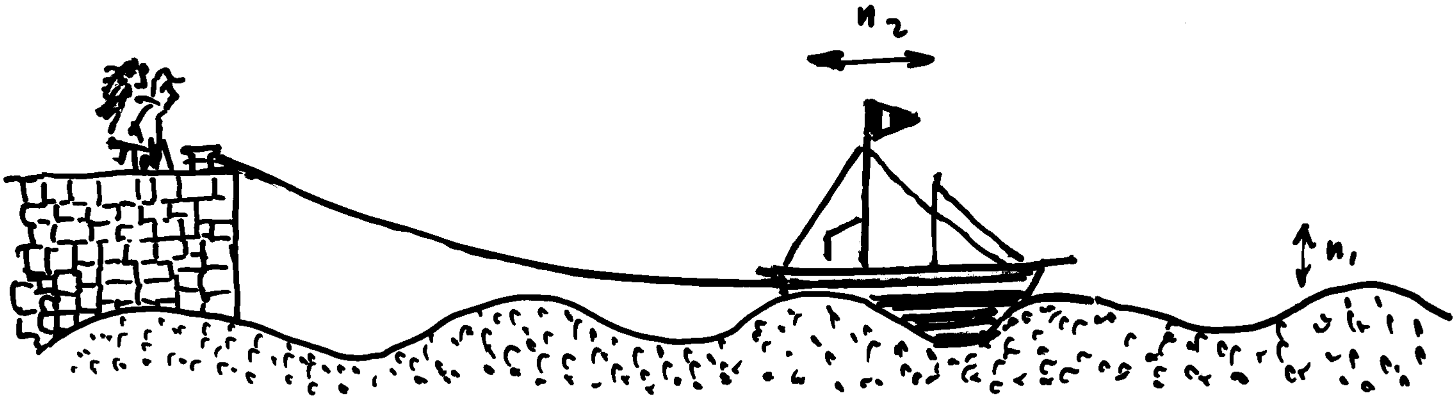
$$\omega_k = k\,\omega$$

$$T_k = \frac{T}{k}$$



subharmonics

Example.



she looks at the waves and counts $\rightarrow n_1$

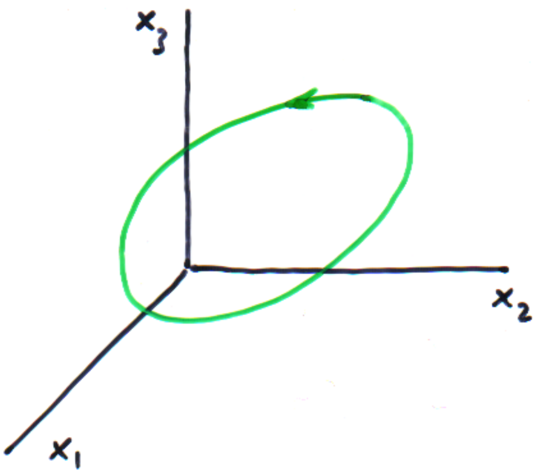
he looks at the boat and counts $\rightarrow n_2$

$$\frac{n_1}{n_2} \simeq \begin{cases} 2 \\ 4 \\ 8 \end{cases} \leftarrow \qquad T_2 \simeq \begin{cases} 2\,T_1 \\ 4\,T_1 \\ 8\,T_1 \end{cases}$$

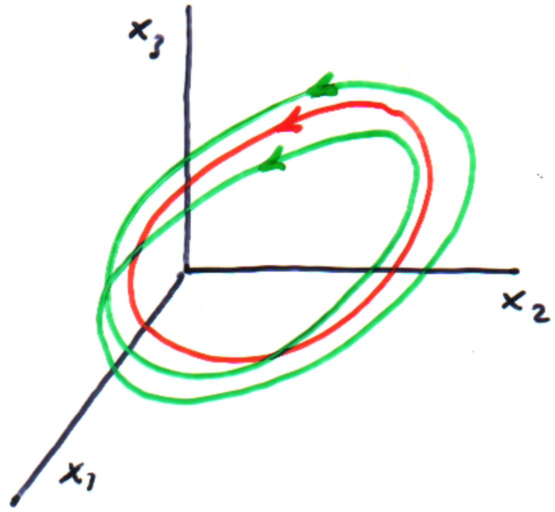
← period doubling

← cascade of period doublings

Flip bifurcation



before $(\bar{p} - \varepsilon)$



after $(\bar{p} + \varepsilon)$

From the left: a stable cycle of period T bifurcates into an unstable cycle of period T and a stable cycle of period $2T$

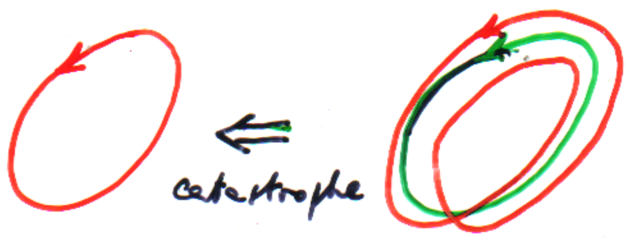
This bifurcation is called flip or "period doubling"

When p is varied further the period and the form of the stable cycle change.

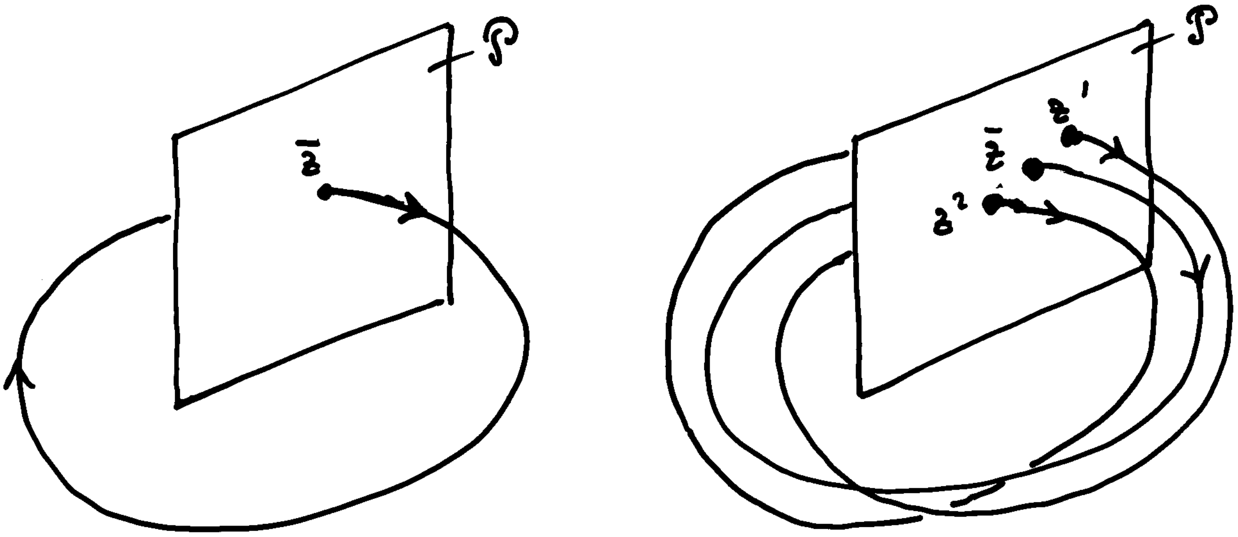
Varying the parameter further there can be other period doublings $T_1 \to 2T_1 \rightsquigarrow T_2 \to 2T_2 \rightsquigarrow T_3 \to 2T_3 \to \dots$

Sometimes a sequence of period doublings is referred to as "period 1, 2, 4, ..."

We can also have catastrophic flips



How can a flip be detected ?



$z(t+1) = P(z(t))$ Poincaré map

The points z^1 and z^2 are very close to $\bar{z}$ and visited alternatively

$\delta^1 = z^1 - \bar{z}$

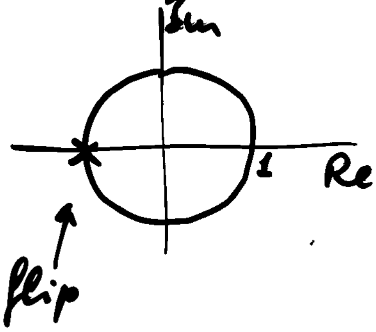
$\delta^2 = z^2 - \bar{z}$

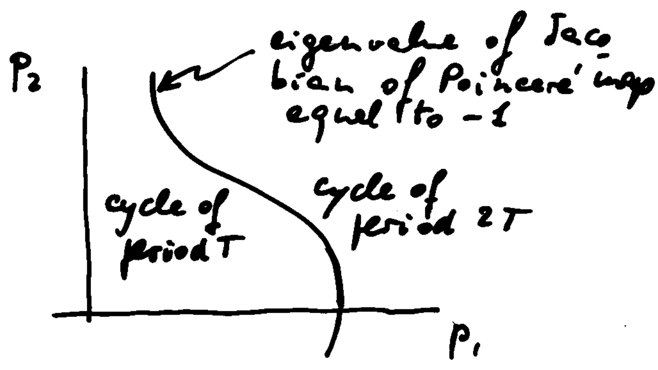
$J\delta^1 = \delta^2 \qquad J\delta^2 = \delta^1$

$J^2\delta^1 = \delta^1$

$\Downarrow$

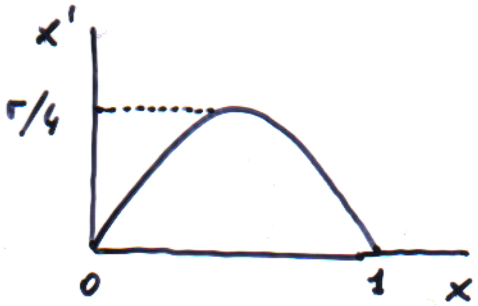
J has an eigenvalue equal to -1





Logistic map : equilibria

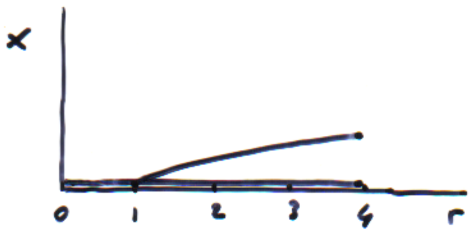
$$x(t+1) = r\,x(t)\big(1 - x(t)\big) \implies x' = r x (1-x)$$



if $r \leq 4$: $[0,1] \to [0,1]$

Equilibria

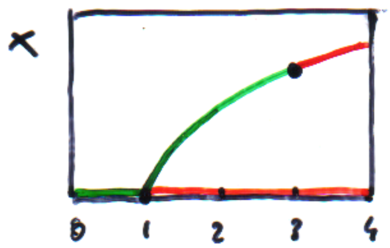
$$x' = x \implies r x^2 + (1-r)x = 0 \implies x = \begin{cases} 0 \\ \frac{r-1}{r} \quad (>0 \text{ if } r>1) \end{cases}$$



Stability

$$\left.\frac{df}{dx}\right|_{x=0} = r \implies x=0 \text{ is stable for } r<1 \text{ and unstable for } r>1$$

$$\left.\frac{df}{dx}\right|_{x=\frac{r-1}{r}} = \dots = 2-r \implies x = \frac{r-1}{r} \text{ is stable for } 1<r<3 \text{ and unstable for } 3<r<4$$



↑ eigenvalue -1 ⇒ flip

Remark : for $3<r<4$ there are no stable equilibria

⇓

"interesting" behavior

Logistic map : Feigenbaum's cascade and chaos

An equilibrium is a period 1 solution
We have already seen that for $r = 3$ there is a period doubling
This means that for $r = 3 + \varepsilon$ we must have a cycle of period 2

$$x_1 \to x_2 \to x_1 \to x_2 \to x_1 \to x_2 \to \cdots$$

x_1 and x_2 are equilibria of the so-called second iterate

$$x_1 = f(f(x_1)) \qquad x_2 = f(f(x_2))$$

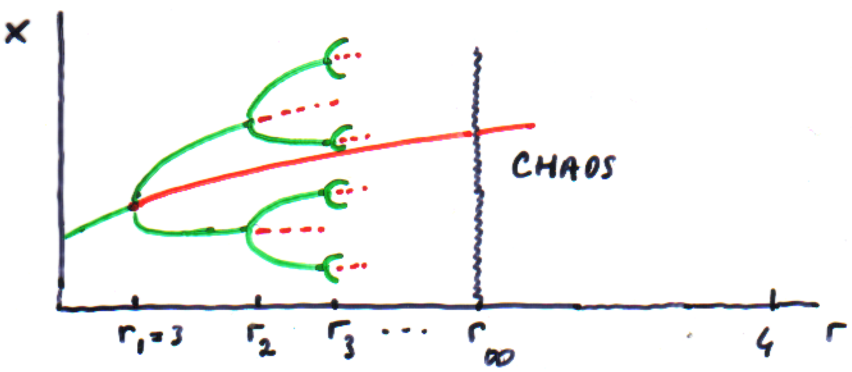
The two solutions of the equation $x = f^{(2)}(x)$ i.e.

$$x = f(f(x))$$

are x_1 and x_2. Finding x_1 and x_2 explicitely is not too difficult.
Extending this method one can look for period 4, 8, ... solutions

$$x = f^{(4)}(x) \quad , \quad x = f^{(8)}(x) \quad , \quad \ldots$$

The result is the following



flips

$$r_\infty = 3.5699\ldots \text{(irrational)}$$

$$\Delta r_n = r_n - r_{n-1}$$

$$\Delta r_{n+1} \underset{n \to \infty}{=} \frac{1}{\delta} \Delta r_n$$

$$\delta = 4.6692\ldots$$

Feigenbaum constant

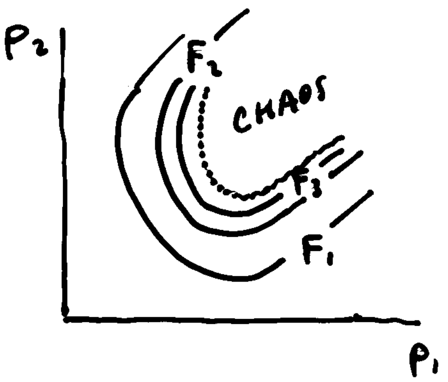
<u>conclusion</u> : an infinite sequence (cascade) of flips announces chaos

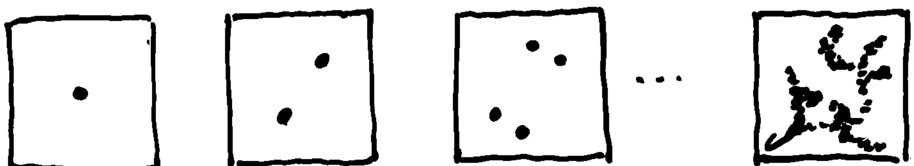
Universality

The Feigenbaum cascade is present and has the same δ (which is, therefore, a universal constant: the π of dynamical systems) in many classes of discrete-time and continuous-time dynamical systems.

For example, in continuous-time systems the following often holds

$$\dot{x}_1 = f_1(x_1\, x_2\, x_3)$$
$$\dot{x}_2 = f_2(x_1\, x_2\, x_3)$$
$$\dot{x}_3 = f_3(x_1\, x_2\, x_3)$$



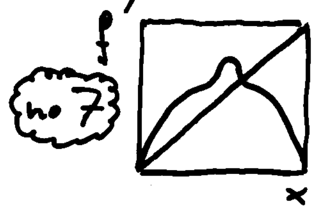


strange attractor

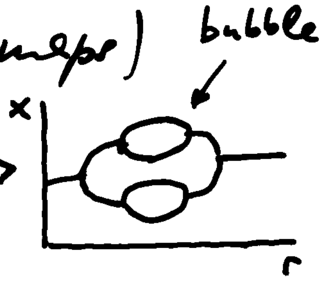
Universality theorem (1-dimensional quadratic maps)

$$x' = f(r, x) \quad \begin{cases} 1.\ \partial f/\partial r > 0 \\ 2.\ f(r\,\cdot) : [0,1] \to [0,1] \\ 3.\ f(r\,0) = f(r\,1) = 0 \\ 4.\ f(r\,\cdot)\ \text{unimodal} \\ 5.\ f''(r, x^*) < 0 \quad \text{where } f'(r\,x^*) = 0 \\ 6.\ f(r,\cdot) \in C^3 \\ 7.\ f'''/f' - \frac{3}{2}\left(f''/f'\right)^2 < 0 \end{cases}$$

negative Schwarzian derivative



$\Rightarrow$



$\Rightarrow$ Feigenbaum cascade