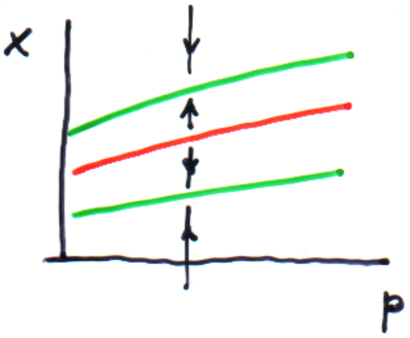
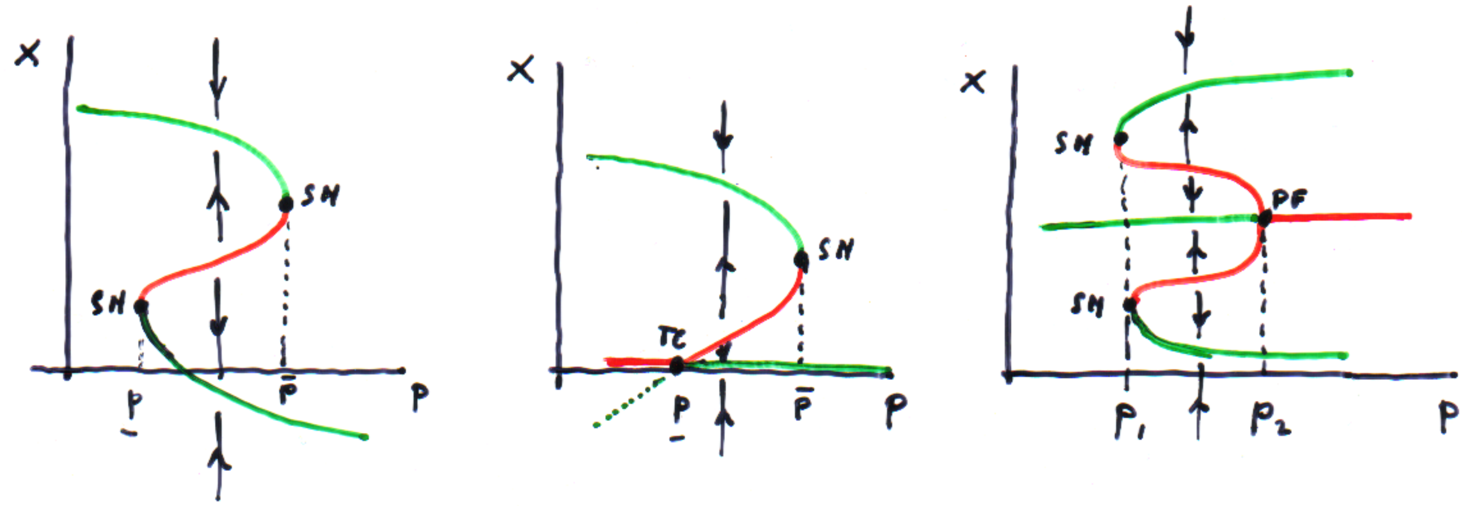
Bistability and hysteresis

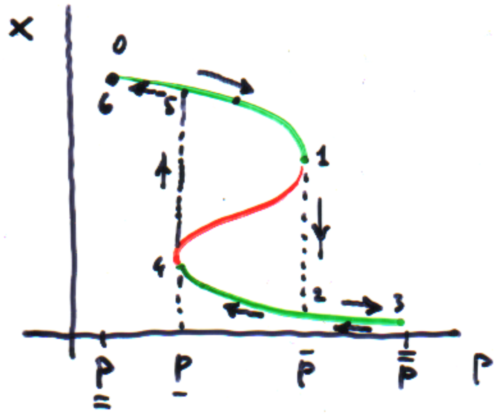


when there are two attractors we say that the system is "bistable"

Bistability is often present in a range of p



What happens if the parameter is varied step-by-step over a range $[\underline{\underline{p}}, \bar{\bar{p}}]$ larger than $[\underline{p}, \bar{p}]$?



each time the parameter is changed the system goes toward a new equilibrium, following the sequence 0, 1, 2, ..., 4, 5, 6=0
The corresponding graph is called hysteresis or hysteretic loop.

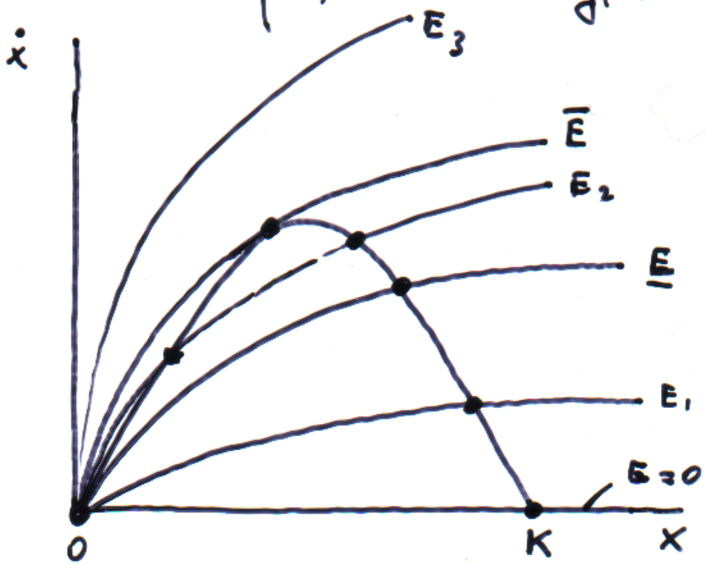
conclusion in order to have hysteresis at least two bifurcations are needed.

A fishery model

$x(t)$ = fish stock

E = fishing effort $\equiv$ # boats

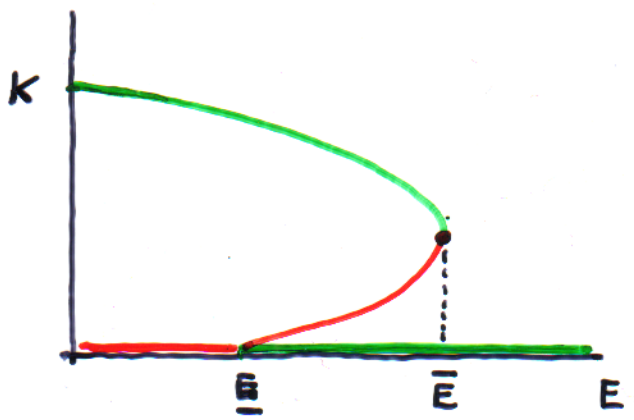
$$\dot{x} = \underbrace{rx\left(1 - \frac{x}{K}\right)}_{f(x)} - \underbrace{\frac{ax}{b+x}}_{g(x)} E$$



For $E = \underline{E}$ $\quad f'(0) = g'(0)E$

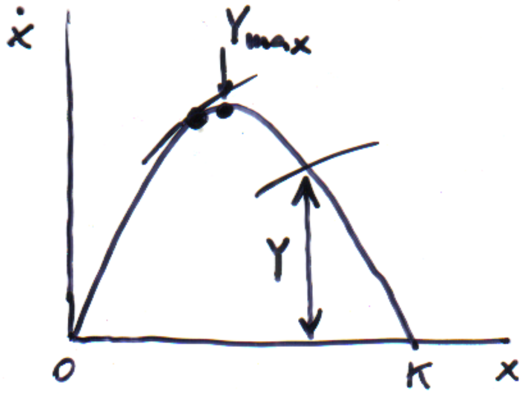
For $E = \bar{E}$ $\quad$ two equilibria collide

For $\underline{E} < E < \bar{E}$ there are three equilibria : the origin and two positive equilibria



The highest equilibrium is stable because in its neighborhood $Eg' \gtrless f'$ if $x \gtrless \bar{x}$

The system has an hysteresis



Y = yield = production rate at equilibrium

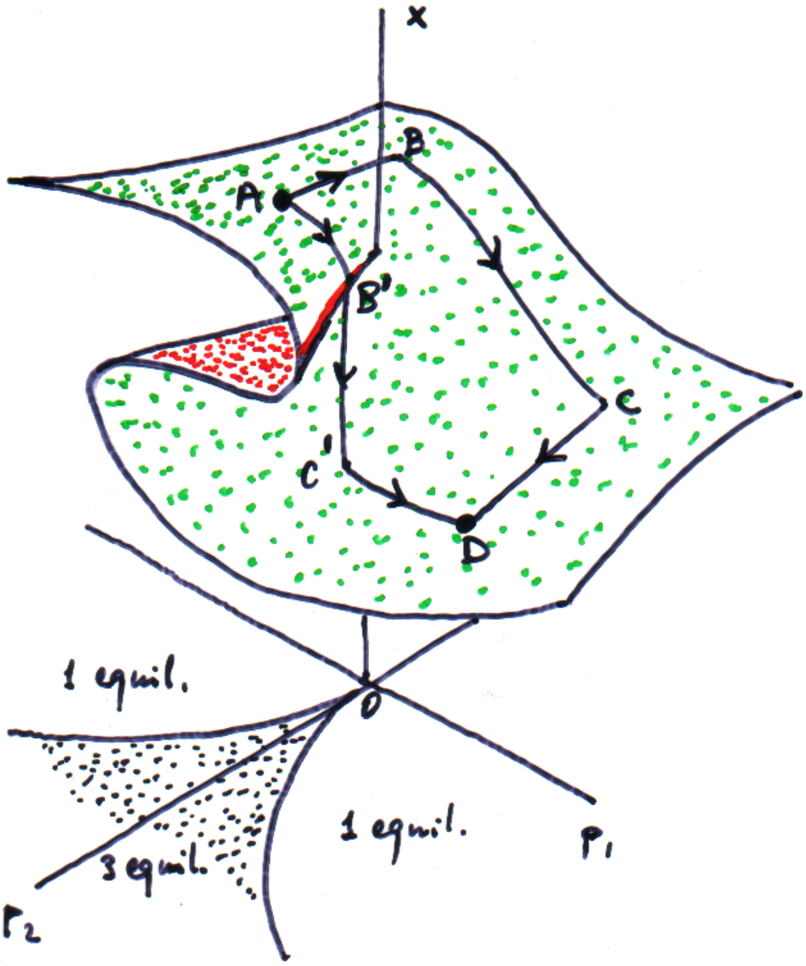
The maximum yield Y_{max} is very close to a catastrophe (the tragedy of natural resources!)

Cusp

The analysis of a dynamical system with two parameters very often points out a structure called "cusp".

The cusp is characterised by 2 saddle-nodes with respect to one parameter and by 1 pitchfork with respect to the other parameter.

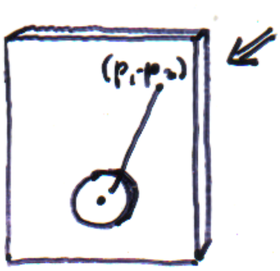
The normal form of the cusp is $\boxed{\dot{x} = p_1 + p_2 x - x^3}$



If $(p_1, p_2) \in$ cusp region there are 3 equil. (2 stable and 1 unstable).
If $(p_1, p_2) \notin$ cusp region there is only 1 equil. (which is stable).

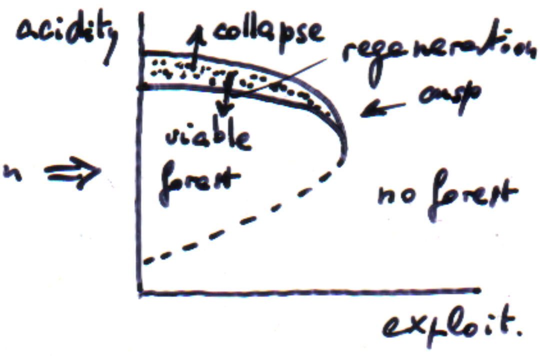
Varying the parameters step-by-step one can bring the system from A to D smoothly. But there are also alternative paths passing through catastrophes

Examples cardiac block



Zeeman's machine

acid rain and forest exploitation ⇒



Problems

P. 1.[1] Describe (in words) a phenomenon that you have observed which is an hysteresis.

P. 2.[2] Discuss the dynamics of the 1-st order system

$$\dot{x} = p x + x^3 - x^5$$

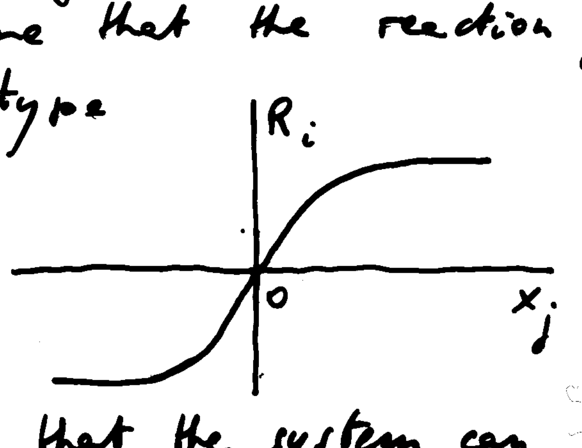
by plotting its equilibria $\bar{x}$ versus p. In particular find out the bifurcations and determine if the system has hysteresis.

P. 3.[3] Consider the following model of a couple

$$\dot{x}_1 = - f_1 x_1 + R_1(x_2) + \rho_1 A_2$$

$$\dot{x}_2 = - f_2 x_2 + R_2(x_1) + \rho_2 A_1$$

where x_i is the feeling of individual i for the partner, R_i is the reaction to the love of the partner, A_i is the appeal of individual i, and f_i and ρ_i are positive parameters (forgetting coefficient and sensitivity to appeal). Assume that the reaction function are of this type



$i \neq j$

Show that the system can have multiple equilibria for small (positive) values of the appeals. Letting $A_1 = A_2 = A$ say if there is an hysteresis w.r.t. A