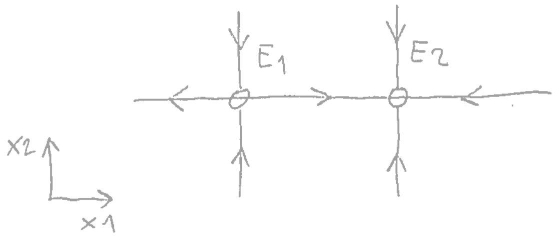
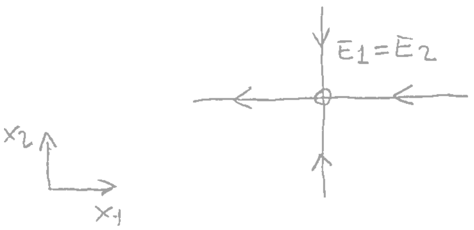
# Algebraic characterization of the transcritical, saddle-node, and pitchfork bifurcations




→ two equilibria, E1 and E2, collide along with change in one parameter $p$

→ before the collision, one eigenvalue, $\lambda_i$, is positive in E1 and negative in E2, otherwise the collision cannot occur

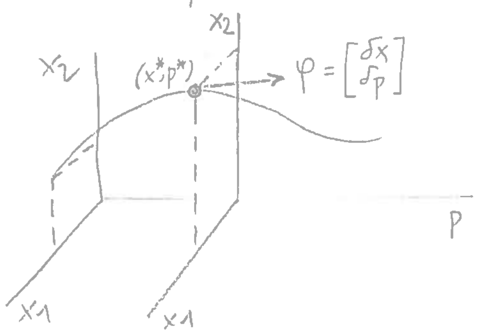



→ the collision occurs in the direction of the eigenvectors associated to $\lambda_i$, that align while approaching the bifurcation

→ at the bifurcation the eigenvalues of E1 and those of E2 coincide, so that

$$\lambda_i = 0 \qquad \text{(condition 1)}$$

The continuation problem $f(x,p)=0$ (for continuous time systems) defines one-dim. equilibrium branches in the $(n+1)$-dim. continuation space $(x,p)$

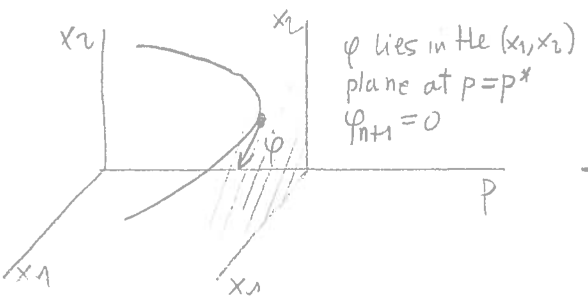



→ expanding $f$ at a point $(x^*,p^*)$ of an equilibrium branch

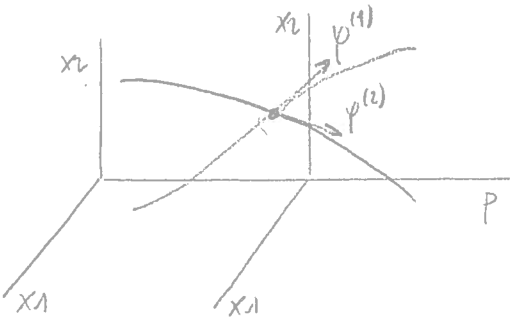
$$f(x,p) = f(x^*+\delta x, p^*+\delta p) =$$

$$= \underbrace{f(x^*,p^*)}_{0} + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \end{bmatrix}_{(x^*,p^*)} \begin{bmatrix} \delta x \\ \delta p \end{bmatrix} + \dots$$

we see that the tangent direction to the eq. branch at $(x^*,p^*)$ belongs to the null-space of the $n\times(n+1)$ Jacobian $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \end{bmatrix}$ evaluated at $(x^*,p^*)$



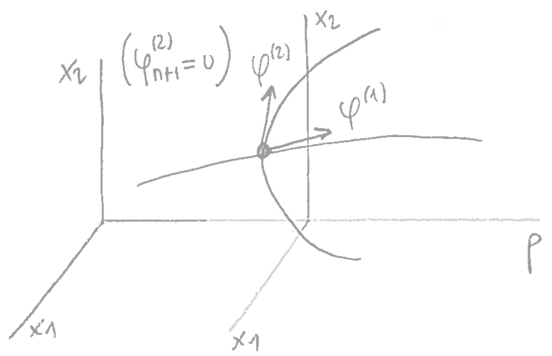

→ if $\operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \end{bmatrix}_{(x^*,p^*)} = n$ (full rank) then the tangent direction is unique.
This is the case of the saddle node bifurcation




→ if $\operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \end{bmatrix}_{(x^*,p^*)} = n-1$ (rank-defect 1, cond. 2) then the null-space $N^*$ of $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \end{bmatrix}_{(x^*,p^*)}$ is 2-dim.

→ generically, there are 2 eq. branches passing through $(x^*,p^*)$ with tangent directions $\varphi^{(1)}, \varphi^{(2)} \in N^*$

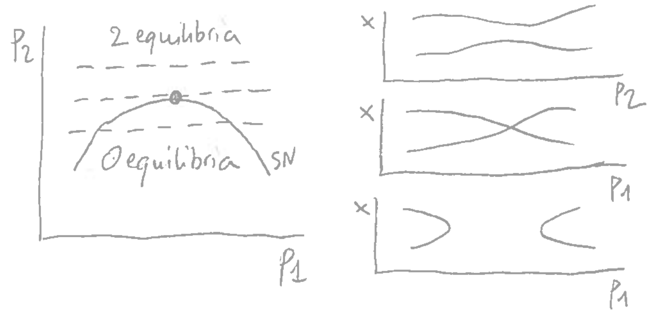
→ if both $\varphi^{(1)}_{n+1}$ and $\varphi^{(2)}_{n+1}$ (the p-components) are nonzero then the bifurcation is transcritical




→ if $\varphi^{(1)}_{n+1} = 0$ or $\varphi^{(2)}_{n+1} = 0$ (condition 3) then the bifurcation is <u>pitchfork</u>
($\varphi^{(1)}_{n+1} = \varphi^{(2)}_{n+1} = 0$ would be a higher codimension)

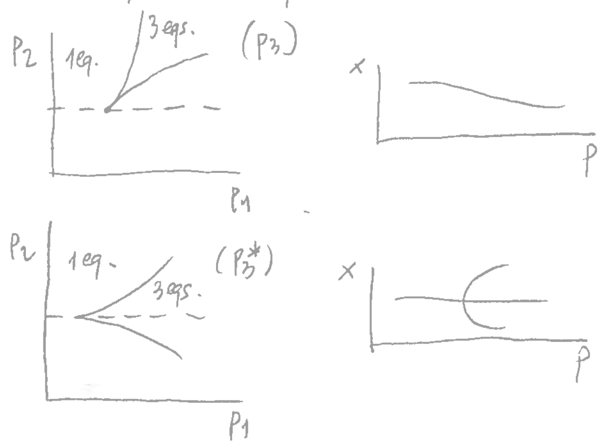
Notes on codimension

→ The transcritical bif. has codim-2.
(two critical conditions 1 + 2)
In fact, it is generically not encountered while moving one par ($p_1$ in the figure), except if another par ($p_2$) is suitably chosen.




→ However, the transcritical has codim-1 when it involves an equilibrium that exists for all values of the parameters, so that it cannot disappear through a SN (e.g. the extinction equilibrium of the logistic model or the equilibrium $(K,0)$ of the prey-predator model). In these situations, condition 2 (the rank-defect of the Jacobian $[\frac{\partial f}{\partial y}\ \frac{\partial f}{\partial p}]$ at the bif) is "built-in" in the system's equations.

→ The pitchfork bif. has codim-3.
(three critical conditions 1+2+3)
In fact, it is generically not encountered while moving one par ($p_1$) even if a second par ($p_2$) is suitably chosen. A third par ($p_3$) should indeed be tuned so that it is possible to enter the cusp while moving $p_1$




→ However, the pitchfork has codim-1 when it involves an equilibrium that exists for all parameter values in a system with a symmetry, so that the equilibrium cannot disappear and the other branch involves two symmetric equilibria that are present on the same side of the bifurcation (e.g. a trivial rest point in a mechanical system that loses stability with the appearance of two new symmetric stable rest points). In these situations conditions 2 and 3 are "built-in" in the system's equations (cond. 2 due to the existence of the trivial equilibrium and cond. 3 due to the symmetry).