Center manifold reduction and normal forms

System (c.t.): $\dot{x} = f(x, p)$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}$

Normal form : $\dot{z} = f_n(z, \alpha)$, $z \in \mathbb{R}^{n_c}$, $\alpha \in \mathbb{R}$

Consider a local bifurcation of the equilibrium $\bar{x}$ at $p = \bar{p}$, with

n_s : # stable eigenvalues ($\mathrm{Re}\,\lambda_i < 0$) , X_s : subspace of associated (standard and generalized) eigenvectors (dim $= n_s$)
n_c : # critical eigenvalues ($\mathrm{Re}\,\lambda_i = 0$) , X_c : ... (dim $= n_c$)
n_u : # unstable eigenvalues ($\mathrm{Re}\,\lambda_i > 0$) , X_u : ... (dim $= n_u$)

stable, center, and unstable manifolds

At $p = \bar{p}$ there are three (locally defined) invariant manifolds passing through $\bar{x}$

stable manifold $\mathcal{X}_s$: dim $= n_s$, tangent to X_s at $\bar{x}$
dynamics on $\mathcal{X}_s$ equivalent to the linearited dyn. on X_s

centermanifold $\mathcal{X}_c$: dim $= n_c$, tangent to X_c at $\bar{x}$
dynamics on $\mathcal{X}_c$ critically dependent on nonlinearities

unstable manifold $\mathcal{X}_u$: dim $= n_u$, tangent to X_u at $\bar{x}$
dynamics on $\mathcal{X}_u$ equivalent to the linearized dyn. on X_u

Parametrized center (or "slow") manifold

For p close to $\bar{p}$ it is possible to define (locally to $\bar{x}(\bar{p})$, the equilibrium $\bar{x}(p)$ might not exist on one side of the bifurcation) a family $\mathcal{X}_c(p)$ of invariant manifolds on which the bifurcation takes place.

$\mathcal{X}_c(p)$: dim $= n_c$, tangent to $X_c(p)$ at $\bar{x}(p)$ (whenever the equilibrium exists)
$X_c(p)$: subspace generated by the (standard and generalized) eigenvectors associated to the λ_i that are critical at the bifurcation.
dynamics on $\mathcal{X}_c(p)$ equivalent to the dynamics of the normal form
(under the change of variable and parameter $x = H(z, \alpha)$, $p = V(\alpha)$,
typically $\bar{x}(\bar{p}) = H(0,0)$, $\bar{p} = V(0)$)

Notes

- the dynamics on $\mathcal{X}_c(p)$ for p close to $\bar{p}$ is "slow" ($|\mathrm{Re}\,\lambda_i|$ small), compared to the dynamics transversal to $\mathcal{X}_c(p)$ ($|\mathrm{Re}\,\lambda_i|$ finitely large)
- the dynamics transversal to $\mathcal{X}_c(p)$ is ruled by th linearited dyn. in the subspace of the non-critical eigenvectors.