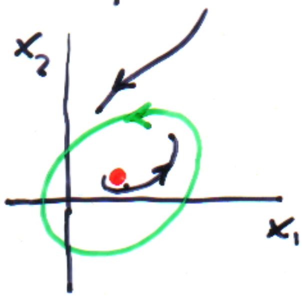
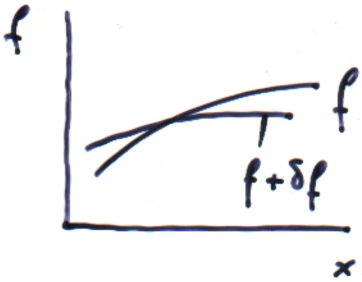
Structurally stable systems

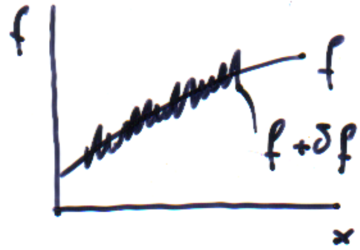
$\dot{x}(t) = f(x(t))$



state portrait

Problem: what happens if f is slightly modified?





?

We will consider the following special, but important, case

$$\dot{x}(t) = f(x(t), p)$$

↳ parameters

and vary the parameters to check if the qualitative behavior of the system changes.

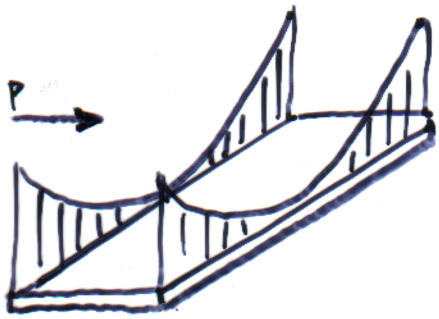
Definition 1 (structural stability)

A system $\dot{x} = f(x, \bar{p})$ is structurally stable iff there exists $\varepsilon > 0$ such that the state portraits of $\dot{x} = f(x, p)$ are topologically equivalent to the state portrait of $\dot{x} = f(x, \bar{p})$

$$\forall\, p : \| p - \bar{p} \| < \varepsilon$$

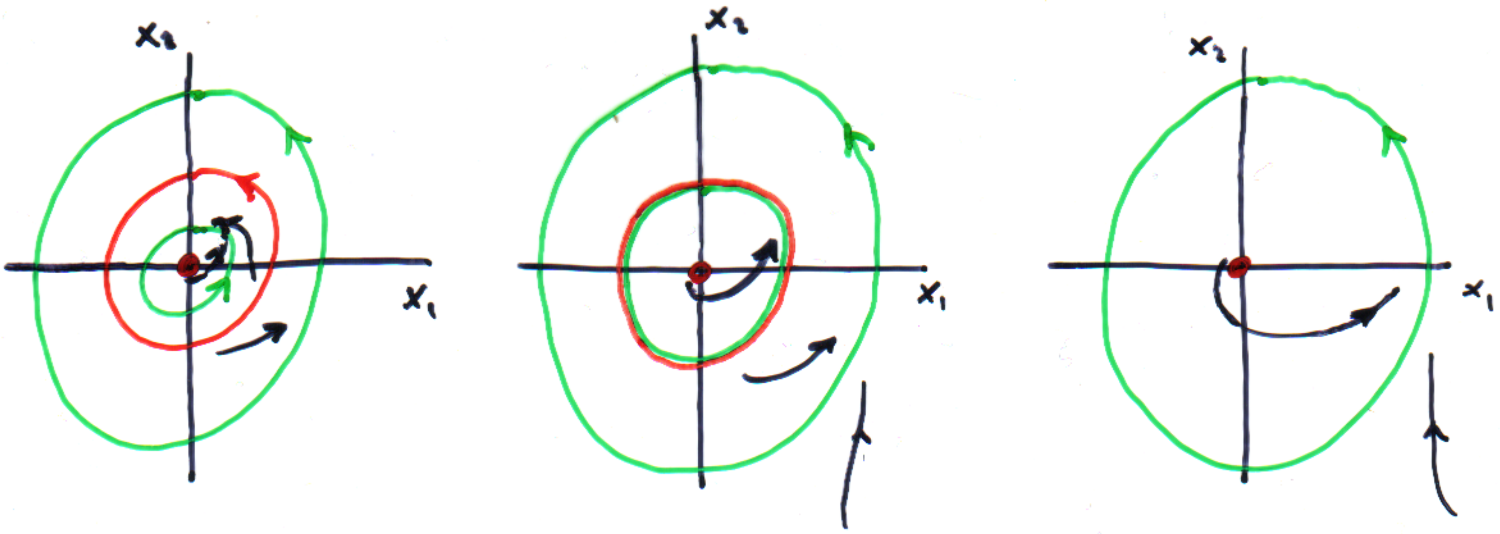
In practice: a small perturbation of the parameters does not change the qualitative behavior of a structurally stable system.

Tacoma's bridge



suspended bridge

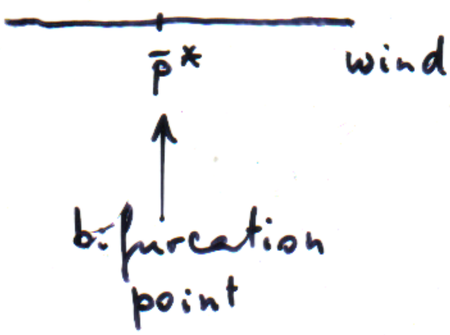
p = wind speed (constant)

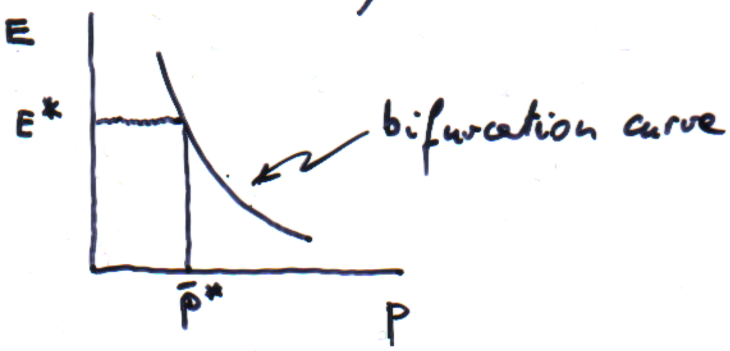


$p = \bar{p}^* - \varepsilon$ $\qquad$ $p = \bar{p}^*$ $\qquad$ $p = \bar{p}^* + \varepsilon$

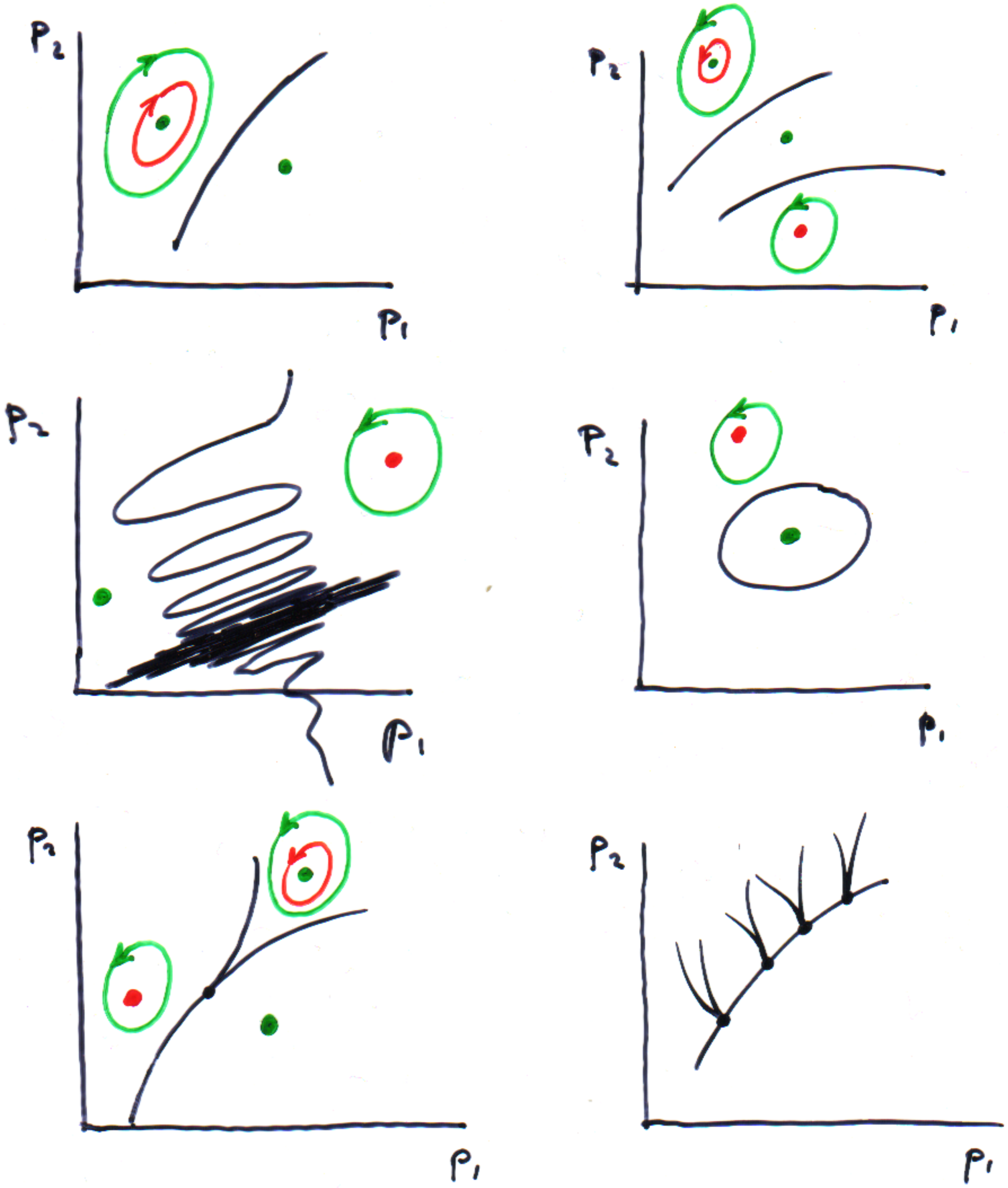
The system with $p = \bar{p}^*$ is not structurally stable

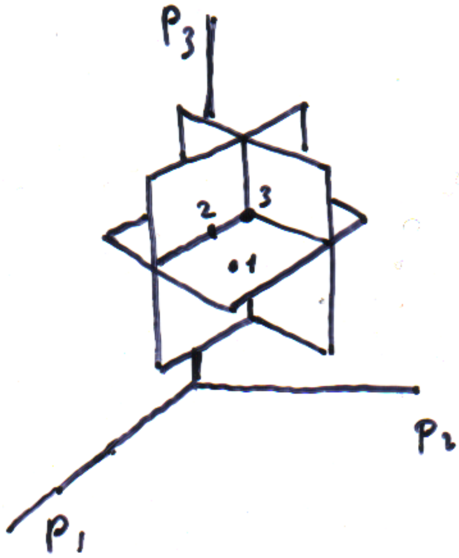
The system for $p = \bar{p}^* - \varepsilon$ or $p = \bar{p}^* + \varepsilon$ is structurally stable





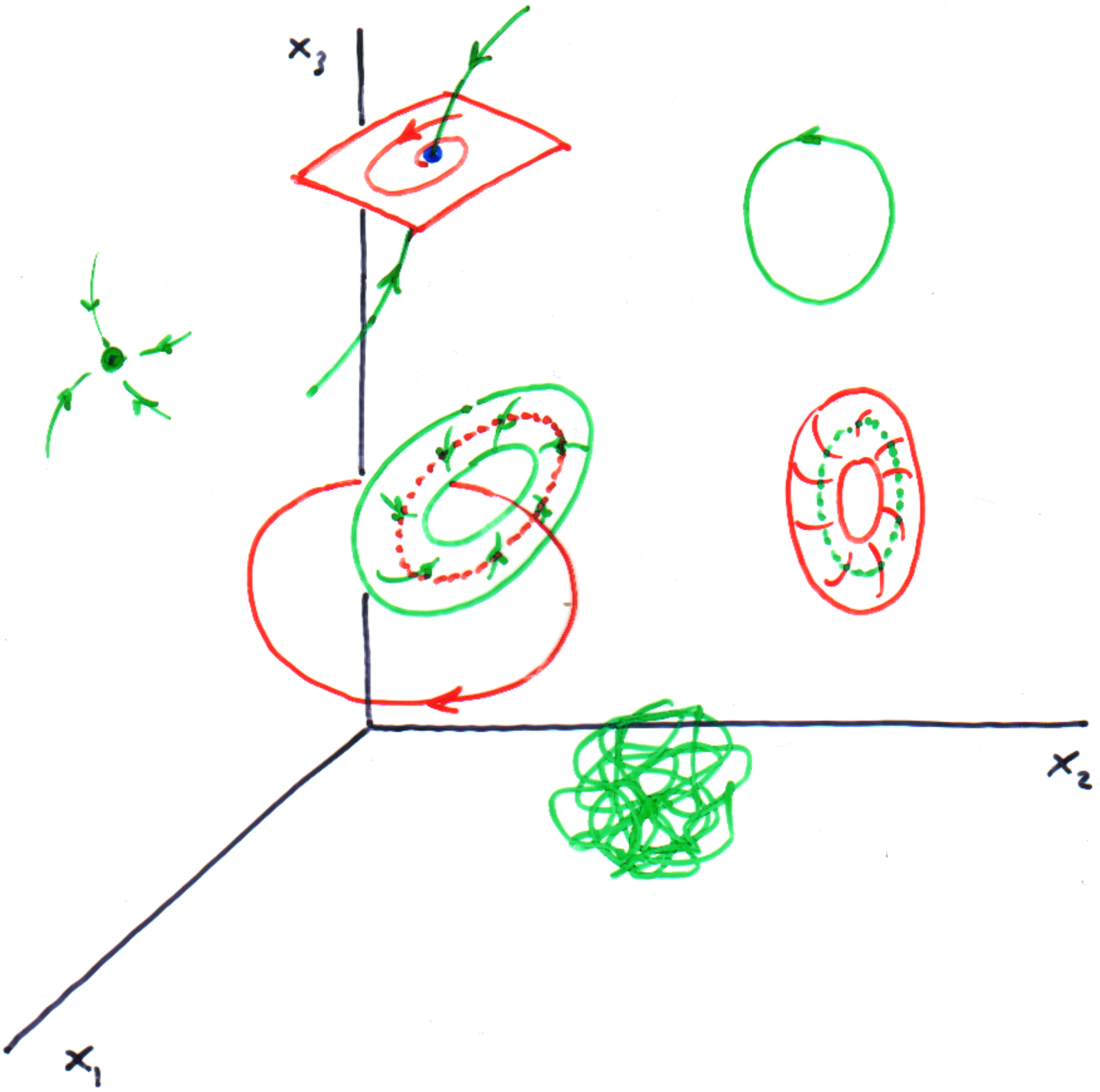
Bifurcation curves





bifurcations of codimension
1, 2, 3

Bifurcations as collisions

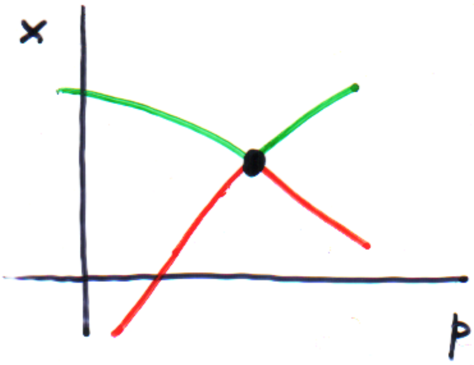


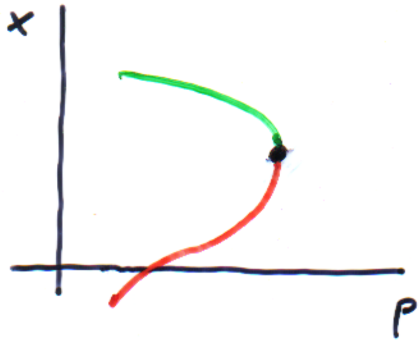
The system is structurally stable if attractors, repellors and saddles (and their stable and unstable manifolds) are "seperated".

In fact, in such a case, a small variations of the parameters implies a small variation of the invariant sets, which remain seperated, so that the portrait of the system remains qualitatively the same

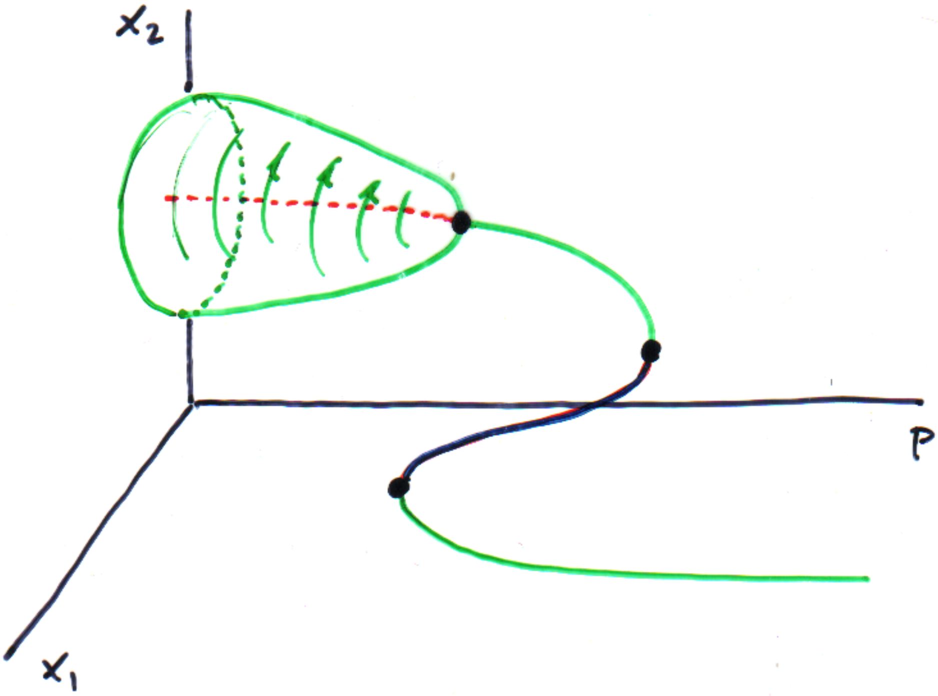
Conclusion bifurcation $\approx$ collision of invariant sets

Bifurcations as collisions

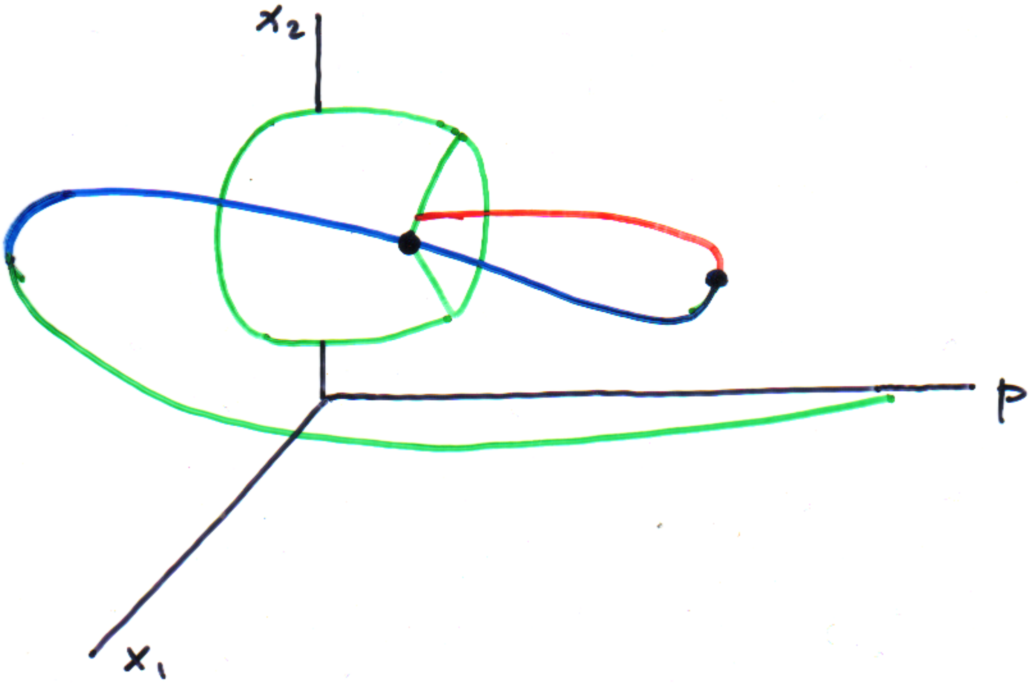




1 state
1 parameter



2 states
1 parameter

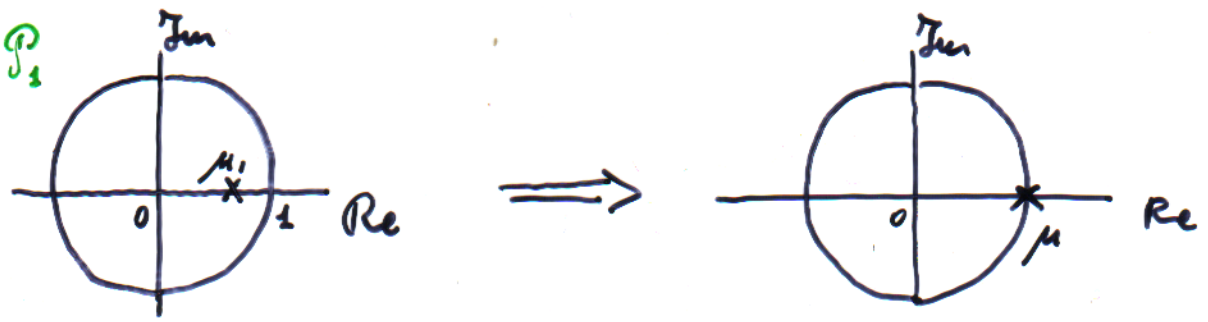


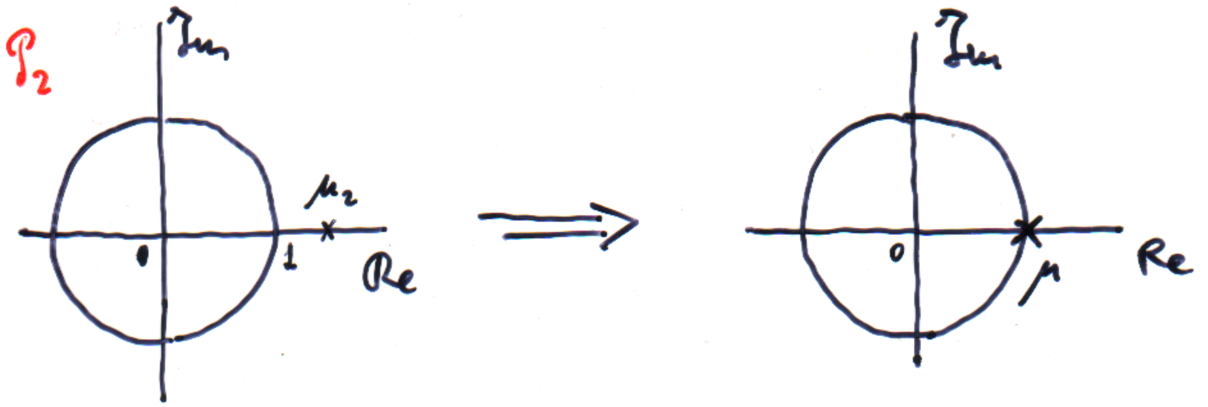
2 states
1 parameter

Local bifurcations

Full collisions of attractors, repellors or saddles

Example

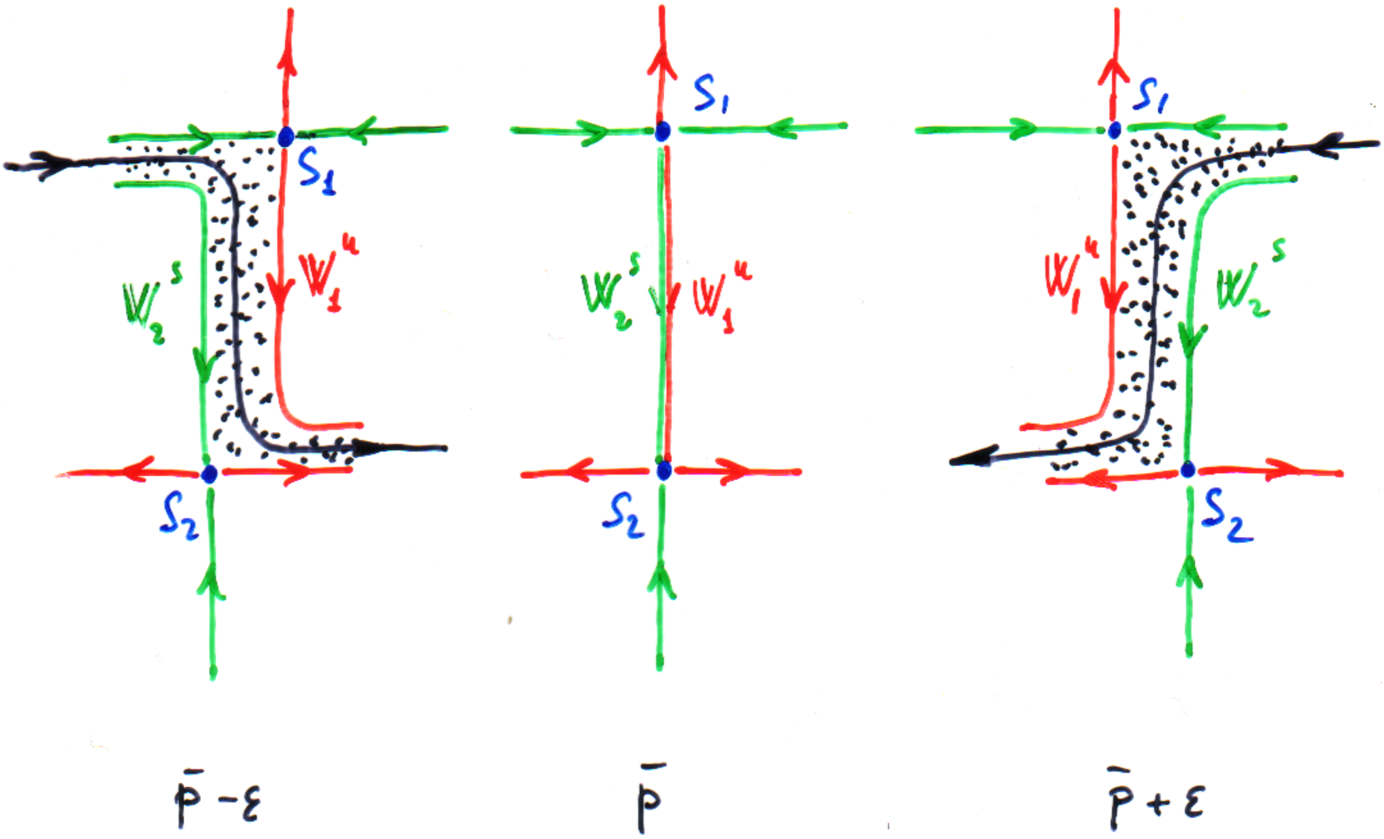




One eigenvalue is on the stability boundary when the collision occurs.

Global bifurcations

Collision of stable and unstable manifolds



At $\bar{p}$ there is a bifurcation because any small perturbation implies a qualitative change of the state portrait.

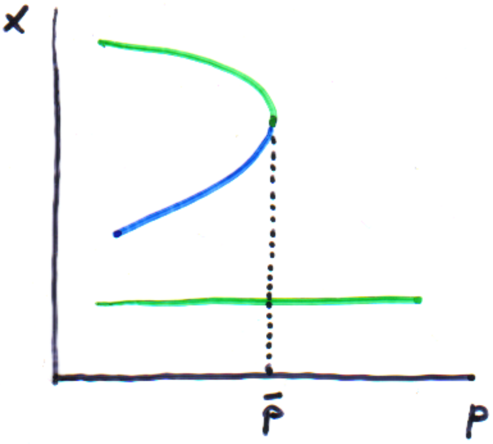
Approaching $\bar{p}$ the manifolds W_1^u and W_2^s become closer and closer and finally collide for $p=\bar{p}$.

For $p=\bar{p}$ there is a saddle to saddle connection.

The bifurcation is not "announced" by an eigenvalue approaching the stability boundary.

These bifurcations are more difficult to detect.

Catastrophic bifurcations



For $p = \bar{p} - \varepsilon$ the system can be in the upper stable equilibrium

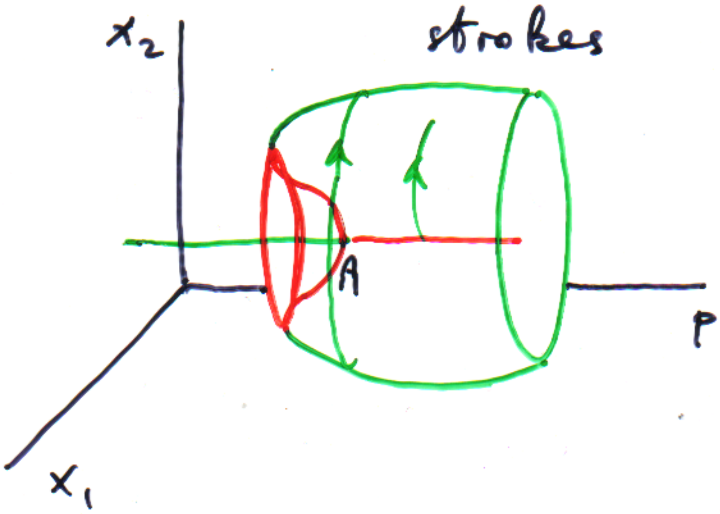
For $p = \bar{p} + \varepsilon$ the system can only be in the lower equilibrium

For a microscopic variation of a parameter we have a macroscopic variation of the equilibrium

In practice for a small variation of the parameter we will have a macroscopic transition from one equilibrium to another.

The macroscopic transition is called catastrophic transition and the bifurcation is called catastrophic

Examples hearthquakes
explosions
revolutions
crashes
strokes



in this case increasing p we have a transition from an equilibrium (A) to a cycle