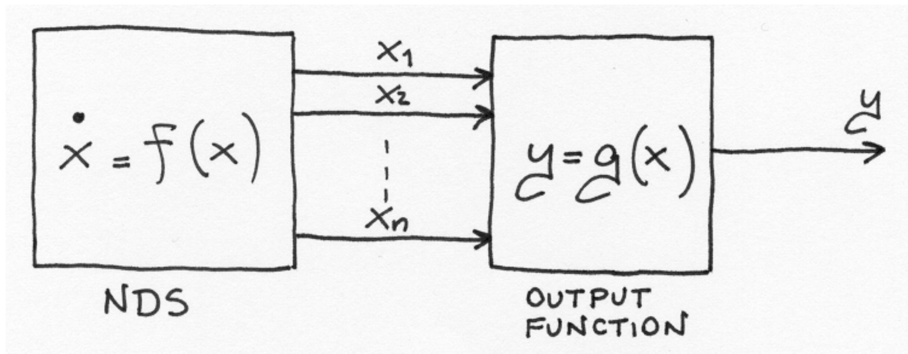
DETERMINISTIC CHAOS

- Time series
- Power spectra
- State-space portraits
- Poincaré sections
- Self-similarity

TIME SERIES

They are obtained by recording the time history of one (or few) of the system variables.

$$\dot{x}(t) = f(x(t))$$
$$y(t) = g(x(t))$$

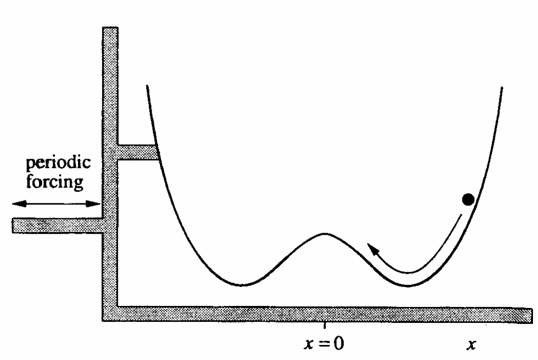


In general, the output is a function of the state variables (often is simply one of them).

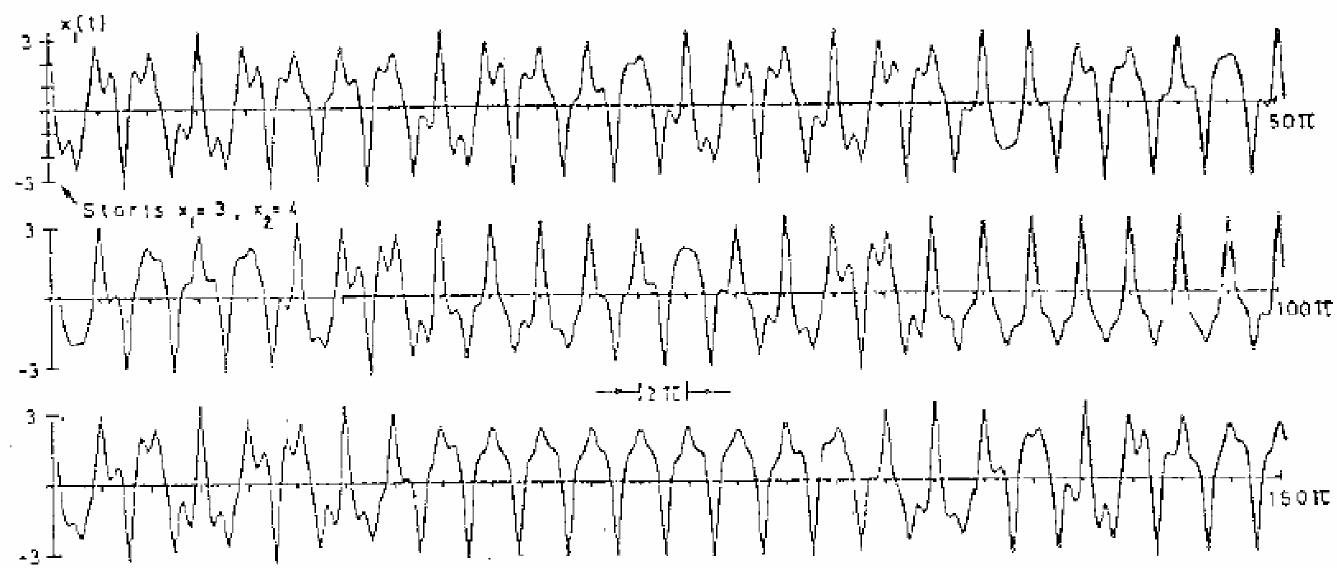
In chaotic regime, $y(t)$ has a non-periodic and apparently random behavior.

Example (continuous-time):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(-F(x_1) - hx_2 + U\sin t)$$

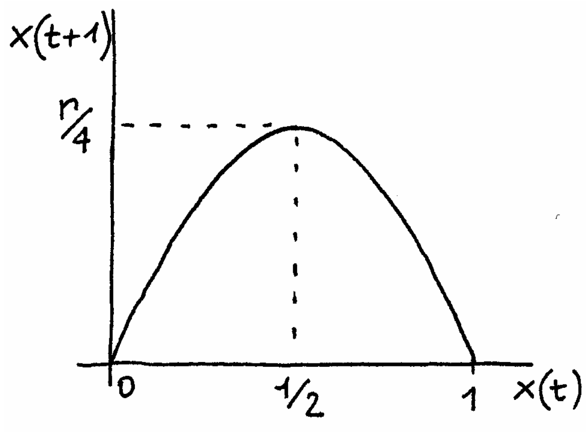


The measured variable is the position: $y(t) = x_1(t)$.

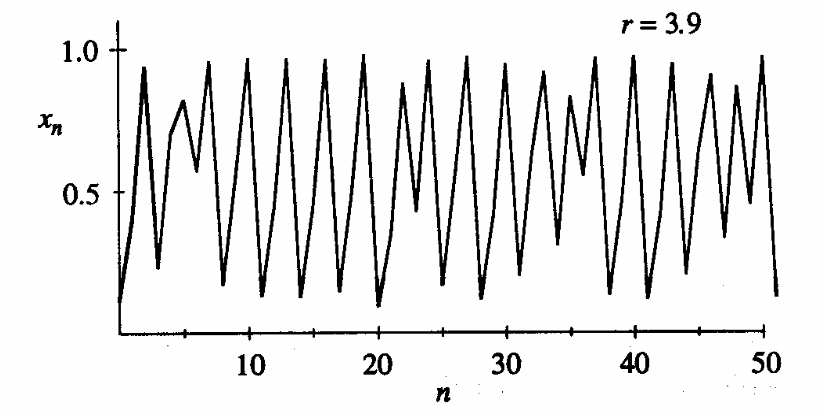


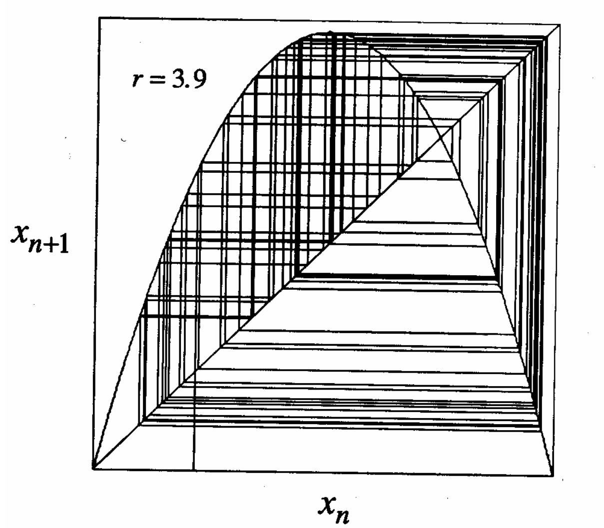
[Example (discrete-time):]() "logistic map":

$$x(t+1) = r\,x(t)\big(1 - x(t)\big)$$



At $r = 3.9$ the behavior of $x(t)$ is non-periodic.





POWER SPECTRA

By means of the Fourier transform, the signal $y(t)$ can be written as

$$y(t) = \frac{1}{\pi}\int_0^{+\infty} Y(\omega)\cos\big(\omega t + \varphi(\omega)\big)d\omega \quad , \quad Y(\omega) > 0$$

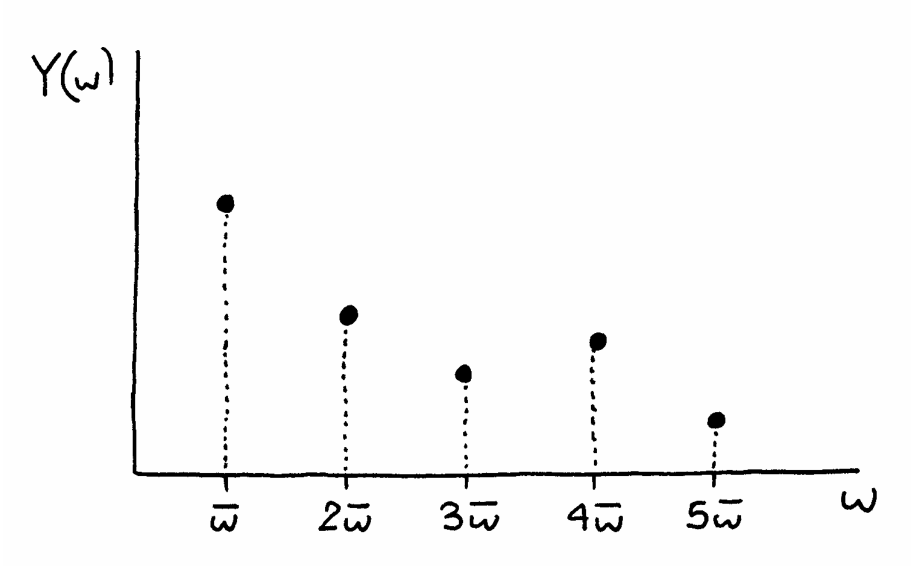
namely as the sum of an infinite number (uncountable, in general) of sinusoidal functions: $Y(\omega)$ is the amplitude of the sinusoid with frequency ω.

The function $Y(\omega)$ is called amplitude spectrum. The function $P(\omega) = Y(\omega)^2$ is called power spectrum.

If $y(t)$ is periodic, with period $T = 2\pi / \varpi$, then:

$$y(t) = Y(0) + \sum_{k=1}^{+\infty} Y(k\varpi)\cos\left(k\varpi + \varphi(k\varpi)\right)$$

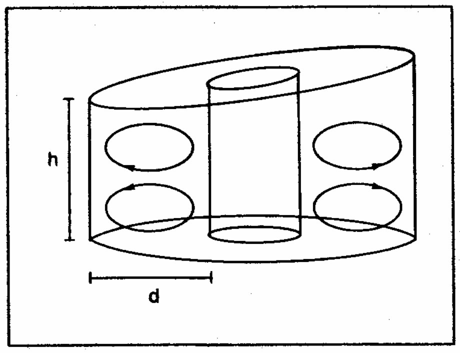
The spectrum is made by "impulses" (= nonzero only when ω is a multiple of ϖ).

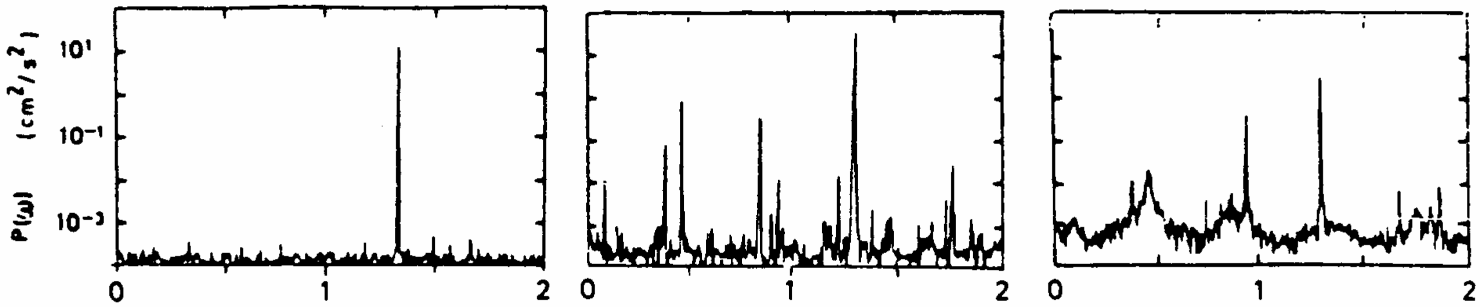


A chaotic signal has typically a "broadband" spectrum.

Example: Taylor-Couette experiment:

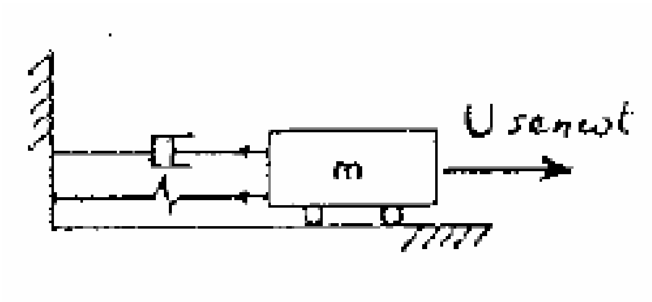
$y(t)$ is the fluid velocity in a given point.

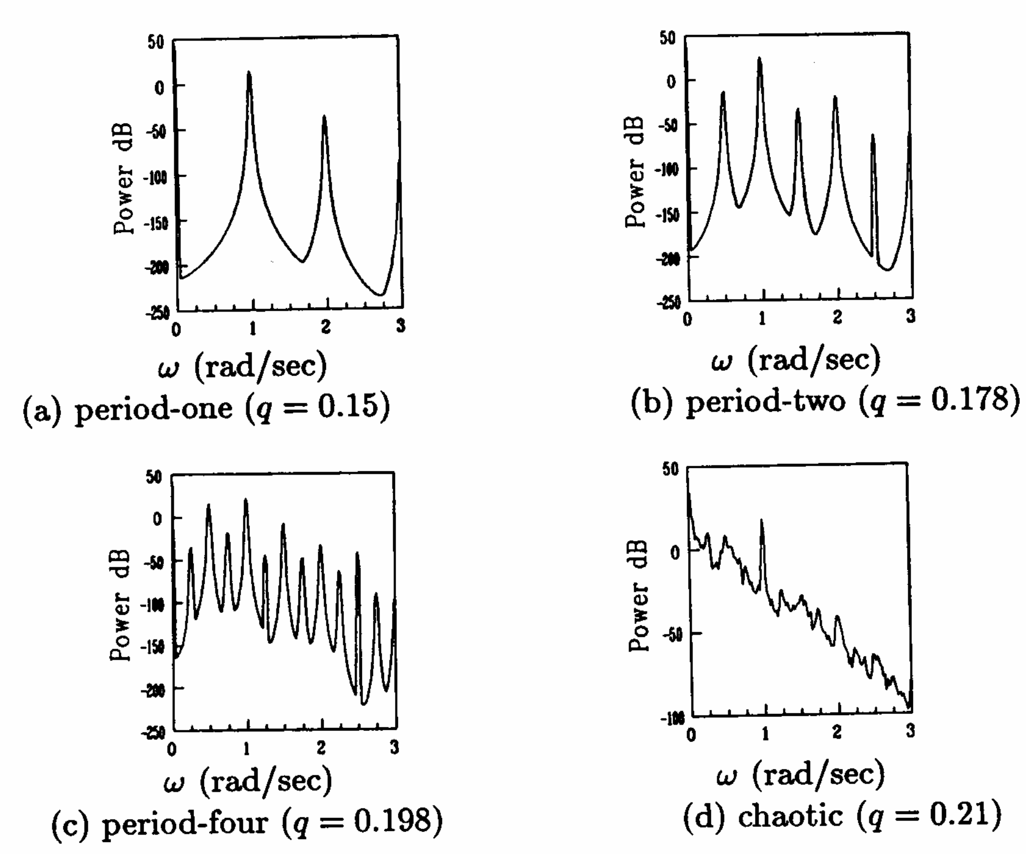




periodic quasi-periodic chaotic

Example: Duffing system:

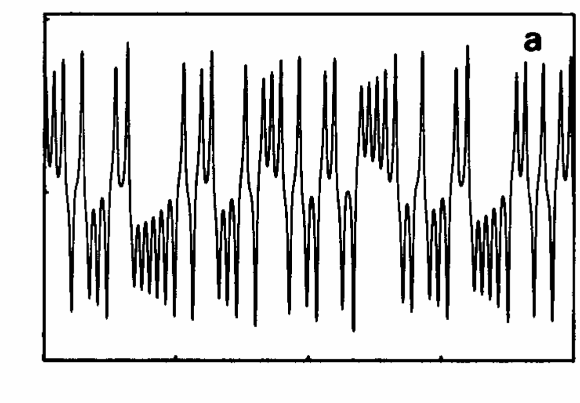




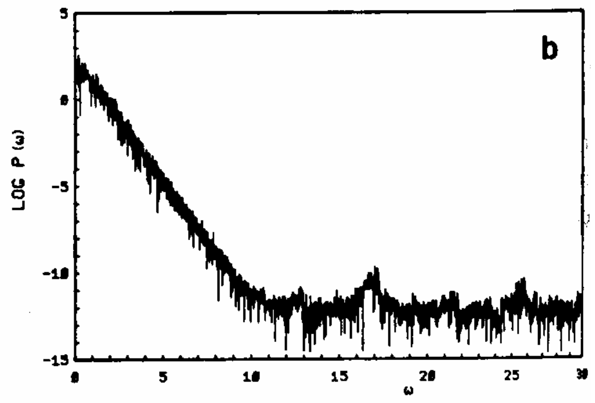
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0.5x_1 - 0.5x_1^3 - 0.168x_2 + q\sin t$$

Example: Lorenz system:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$



time series $x(t)$



power spectrum

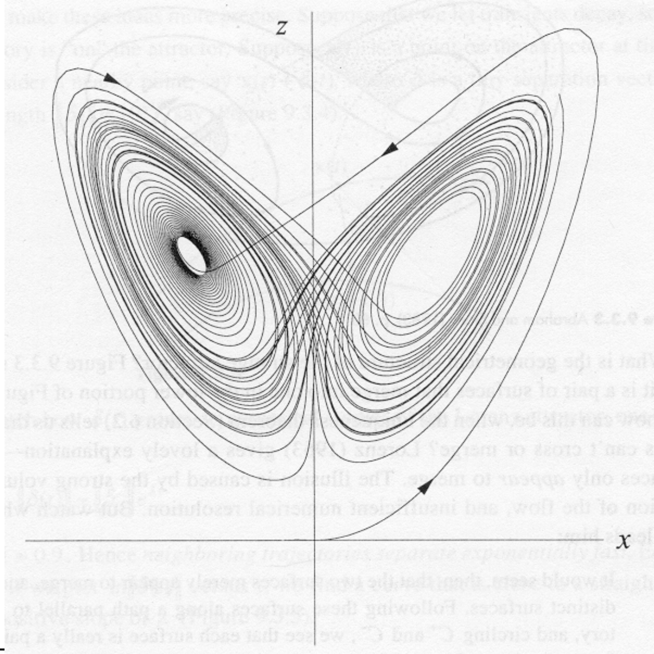
STATE-SPACE PORTRAITS

In chaotic regime, the trajectories of the system

- remain bounded
- never return to a state already visited (=non periodicity), but pass arbitrarily close to it
- display complex geometries

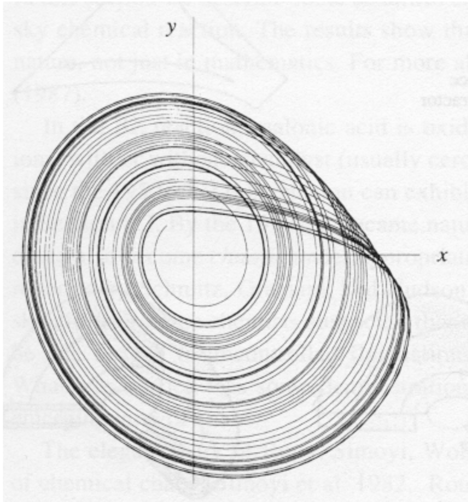
Example: Lorenz system

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$

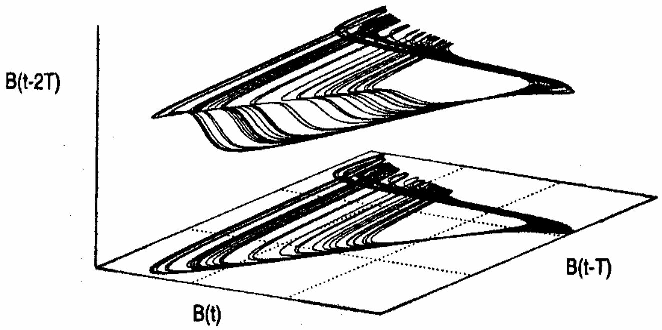


Example: Rössler system

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + (x - c)z$$



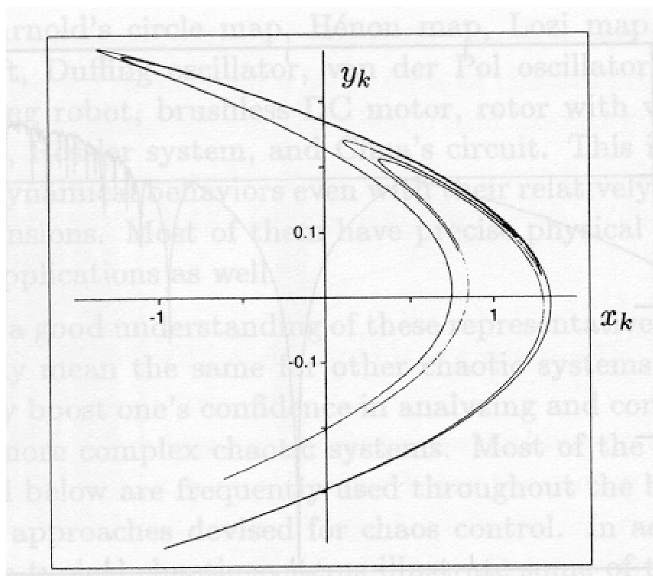
Example: a trajectory "reconstructed" from a time series obtained by an experiment (a chemical reaction)



Example: Henon map (discrete-time system)

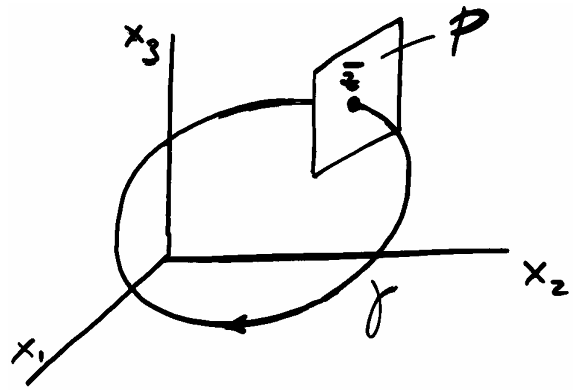
$$x(t+1) = y(t) + 1 - ax(t)^2$$
$$y(t+1) = bx(t)$$

In the state space (x, y), the trajectory is the sequence of points $\big(x(t), y(t)\big)$, $t = 0,1,\ldots$.

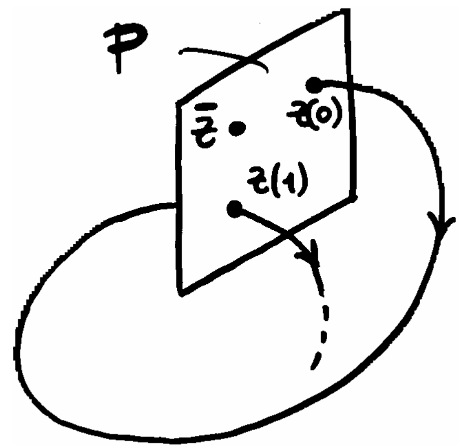


POINCARE' SECTIONS

In a *continuous-time* n-order system $\dot{x} = f(x)$, the Poincaré section is a $(n-1)$-dimensional surface *P*, which is transversal (at a point $\bar{z}$) to a limit cycle γ.



The trajectory started at $z(0) \in P$ will intersect *P* at points $z(1), z(2), \ldots$.



Thus $\dot{x} = f(x)$ defines (close to γ) a *discrete-time* system (Poincaré map)

$$z(t+1) = P(z(t))$$

where $z \in R^{n-1}$, $\bar{z} = P(\bar{z})$.

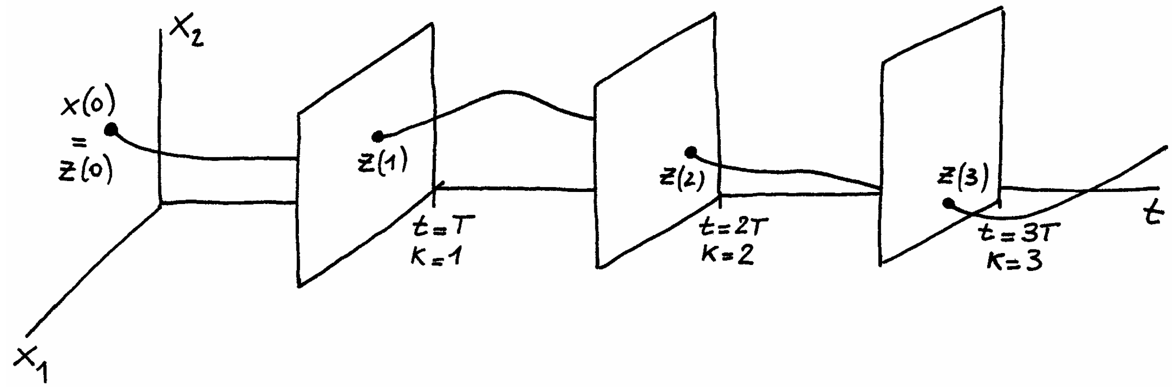
The Poincaré map can also be defined for a continuos-time system $\dot{x}(t) = f(t, x(t))$ which is periodic with respect to t (with period $T > 0$):

$$f(t, x) = f(t+T, x), \text{ for all } t, x$$

We need to consider the period-T map (or "stroboscopic map"):

$$z(k+1) = P(z(k))$$

where $z(k) = x(kT)$, $k = 0,1,2,\ldots$.

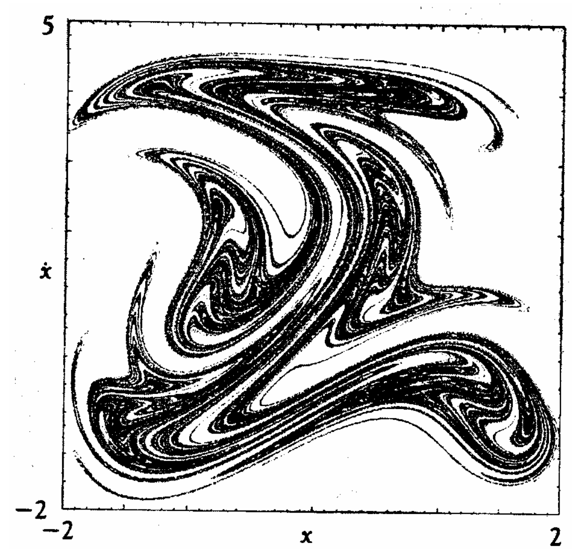


On the Poincaré section, we observe the trajectory of the discrete-time system

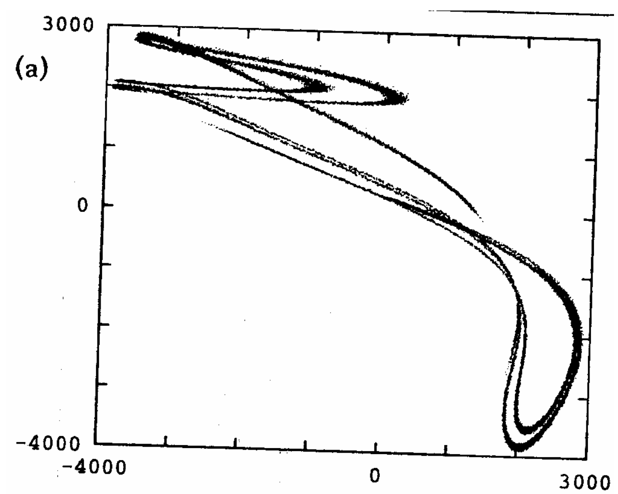
$$z(k+1) = P(z(k))$$

In chaotic regime, on the Poincaré section we observe a bounded set with complex geometry.

Example: potential wells with periodic forcing.



Example: laser: experimentally derived Poincaré section.

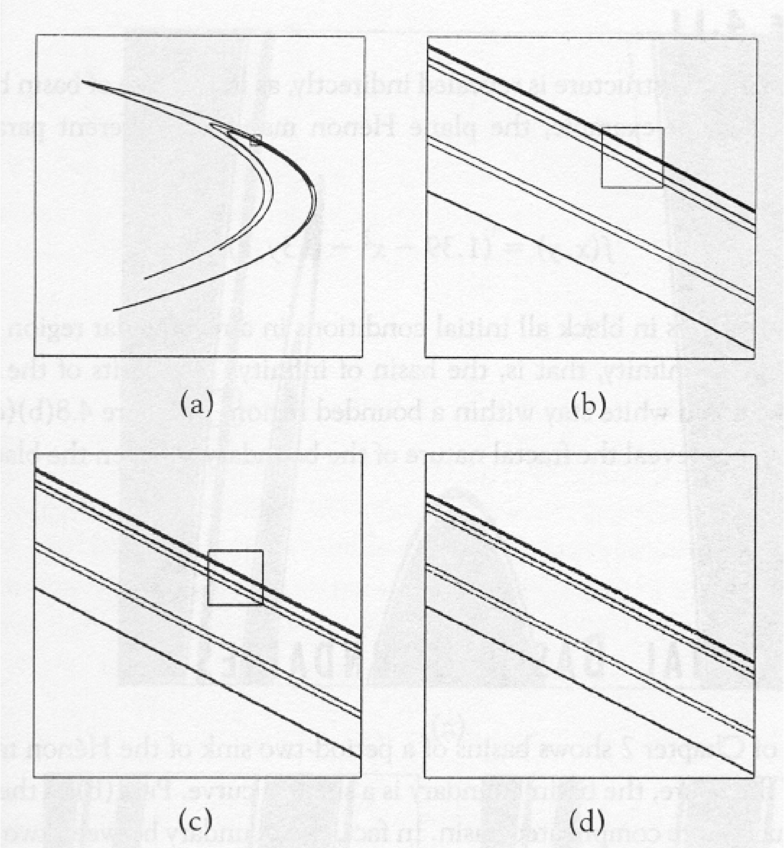


SELF-SIMILARITY

In chaotic regime, the system trajectories have "self-similar" geometry: the same structure is reproduced at arbitrarily small scale.

Example: "zooming" into a trajectory of the Henon map.

The "6-band" structure is repeated infinitely many times.



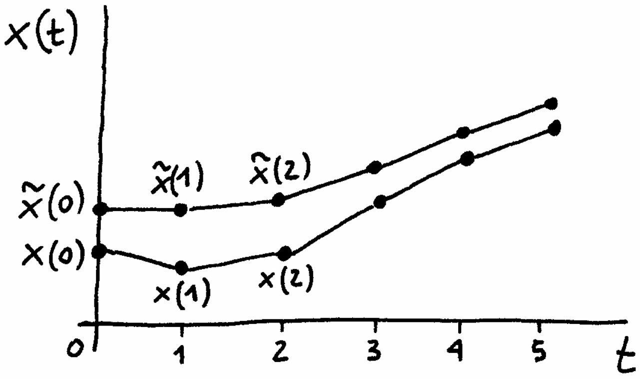
LIAPUNOV EXPONENTS (LEs)

- Discrete-time systems (1- and n-dimensional)
- Continuous-time systems

1-DIMENSIONAL MAPS

Consider

- the discrete-time system $x(t+1) = f(x(t))$, with $n = 1$
- a "nominal" trajectory $\{x(0), x(1), x(2), \cdots\}$
- a "perturbed" trajectory $\{\tilde{x}(0), \tilde{x}(1), \tilde{x}(2), \cdots\}$ started from a state $\tilde{x}(0) = x(0) + \partial x(0)$ "close" to $x(0)$



Since

$$\begin{aligned}\tilde{x}(1) - x(1) &= f(\tilde{x}(0)) - f(x(0)) \\ &= f'(x(0))(\tilde{x}(0) - x(0)) + \ldots\end{aligned}$$

it follows that $|f'(x(0))|$ is the expansion/contraction rate of the initial difference $\partial x(0)$ between the two trajectories (if infinitesimal).

After t time steps

$$\tilde{x}(t) - x(t) = f^t(\tilde{x}(0)) - f^t(x(0)) = \left[\frac{\partial f^t}{\partial x}\right]_{x(0)} \left(\tilde{x}(0) - x(0)\right) + \ldots$$

$$= \left\{ f'\left(x(t-1)\right) f'\left(x(t-2)\right) \cdots f'\left(x(0)\right) \right\} \left(\tilde{x}(0) - x(0)\right) + \ldots$$

Thus, asymptotically the average separation rate (per step) of nearby trajectories is

$$h_{x(0)} = \lim_{t \to \infty} \left| f'\left(x(t-1)\right) f'\left(x(t-2)\right) \cdots f'\left(x(0)\right) \right|^{1/t}$$

If $\partial x(0)$ is infinitesimal, for $t \to \infty$ we have $\left|\partial x(t)\right| \to (h_{x(0)})^t \left|\partial x(0)\right|$ or, equivalently

$$\left|\partial x(t)\right| \to e^{L_{x(0)}\, t} \left|\partial x(0)\right|$$

$L_{x(0)}$ is the Liapunov exponent (LE) of the trajectory started at $x(0)$.

To summarize, the LE is given by

$$L_{x(0)} = \lim_{t \to \infty} \frac{\ln\left|f'\left(x(t-1)\right)\right| + \ln\left|f'\left(x(t-2)\right)\right| + \ldots + \ln\left|f'\left(x(0)\right)\right|}{t}$$

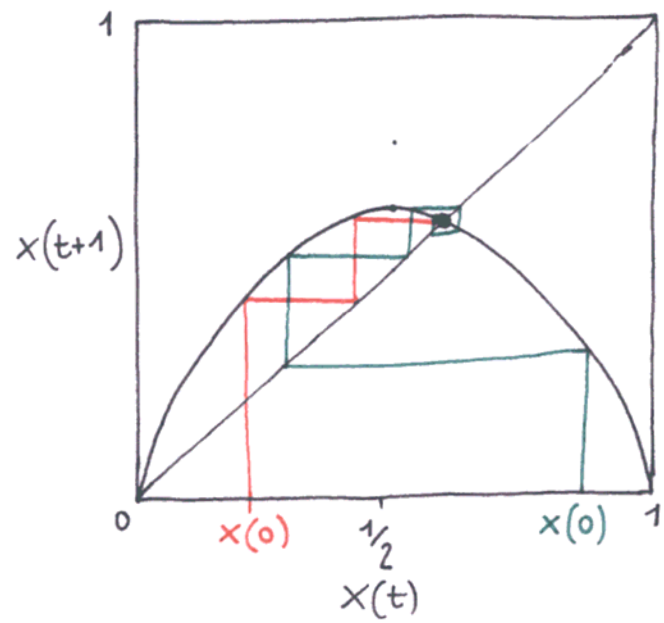
- If $L_{x(0)} > 0$: along the trajectory γ started at $x(0)$, nearby trajectories diverge (on the average) from γ.

- If $L_{x(0)} < 0$: along the trajectory γ started at $x(0)$, nearby trajectories converge (on the average) to γ.

Example: logistic map, $x(t+1) = r\,x(t)\big(1 - x(t)\big)$

At $r = 2.5$, the equilibrium $\bar{x} = (r-1)/r$ is asymptotically stable, because the Jacobian is

$$f'(x) = r - 2r\bar{x} = 2 - r = -0.5$$



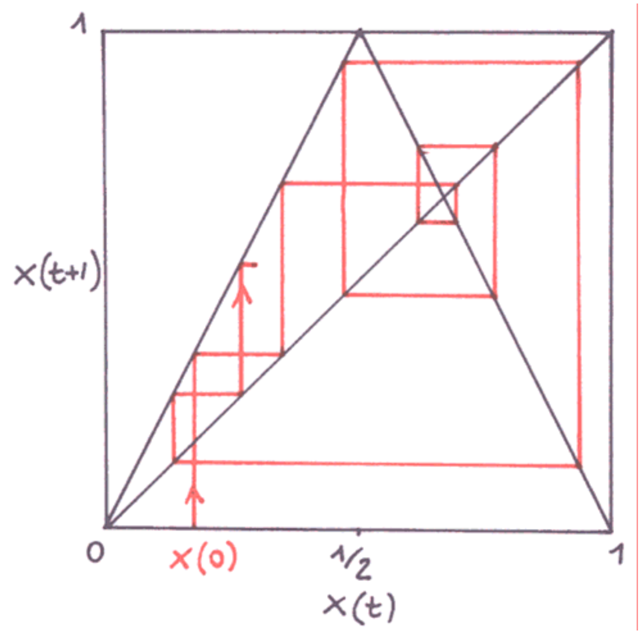
Any trajectory started at $x(0) \in (0,1)$ tends to $\bar{x}$. Then

$$L_{x(0)} = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln\left| f'\big(x(k)\big)\right| = \ln\left| f'(\bar{x})\right| = \ln 0.5 < 0$$

Remark: $\left|f'(\bar{x})\right| < 1$ ($\bar{x}$ asymptotically stable) $\Rightarrow L_{x(0)} < 0$

Example: tent map

$$x(t+1) = \begin{cases} 2x(t) & if\ x(t) \le 1/2 \\ 2\big(1 - x(t)\big) & if\ x(t) > 1/2 \end{cases}$$



The trajectory neither tends to an equilibrium nor to a cycle, but remains non periodic forever.

If we exclude all trajectories passing through $x = 1/2$ (i.e. a zero-measure set of initial states), any $x(0) \in (0,1)$ implies

$$L_{x(0)} = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln\left| f'\big(x(k)\big)\right| = \ln\left| f'(x)\right| = \ln 2 > 0$$

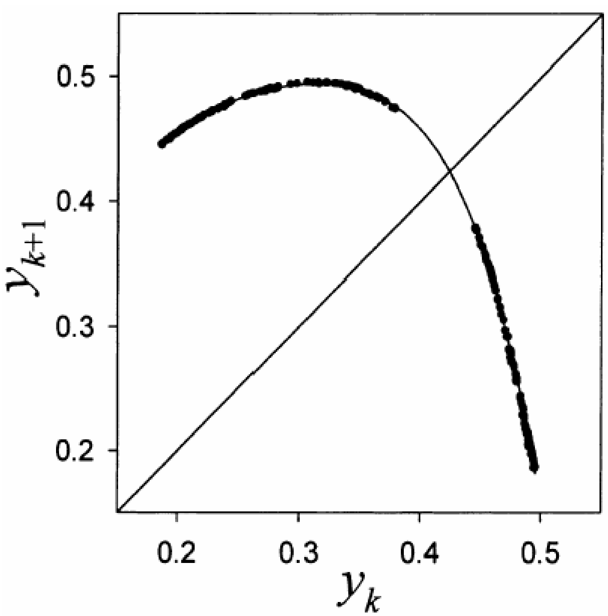
When the trajectory is non-periodic, typically the LE can only be computed numerically.

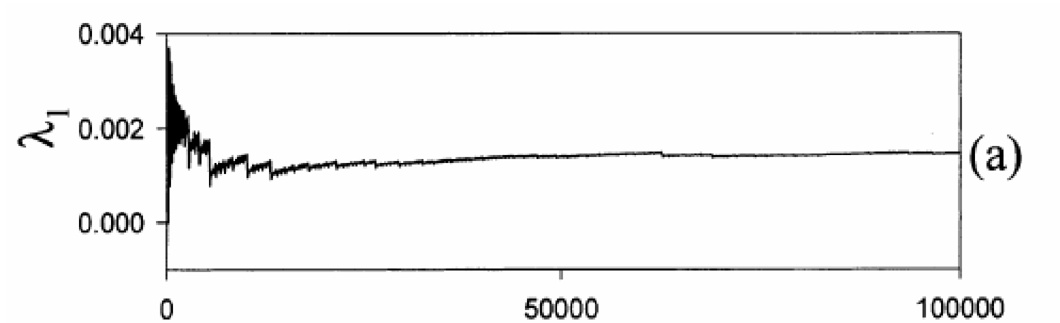
The trajectory $\{x(0), x(1), x(2), \cdots\}$ is recursively obtained at, at the same time, we compute the estimate

$$\hat{L}_{t,x(0)} = \frac{1}{t}\sum_{k=0}^{t-1} \ln\left| f'(x(k))\right|$$

until it converges as t grows.

Example:





The estimate of the Liapunov exponent converges, as t grows, to a positive value.

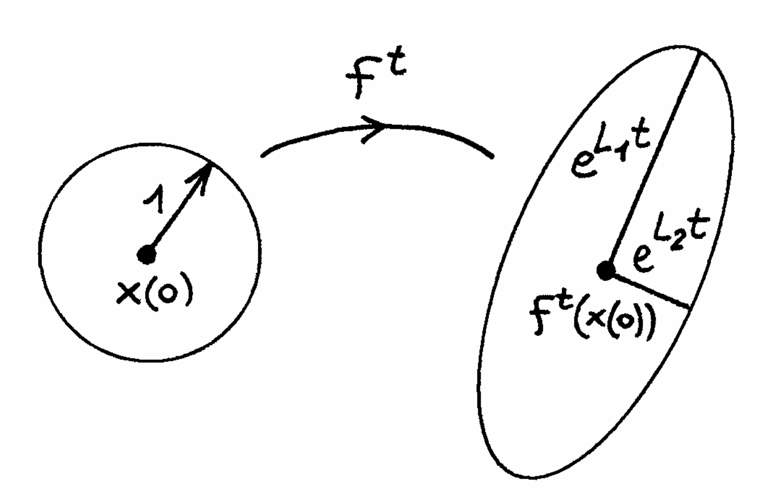
n-DIMENSIONAL MAPS

Consider now a discrete-time system $x(t+1) = f(x(t))$, with any order $n \geq 1$. The LEs are a set of n real numbers, conventionally in decreasing order:

$$L_{1,x(0)} \geq L_{2,x(0)} \geq \cdots \geq L_{n,x(0)}$$

The quantity $\exp(L_{i,x(0)})$ is the growing rate of the distance from the nominal trajectory (=started at $x(0)$) along n orthogonal directions.

- $\exp(L_{1,x(0)})$: "maximum grow" direction (#1)
- $\exp(L_{2,x(0)})$: "maximum grow" direction among those orthogonal to #1 (#2)
- $\exp(L_{3,x(0)})$: "maximum grow" direction among those orthogonal to #1 and #2 (#3)
- ...and so on up to n.

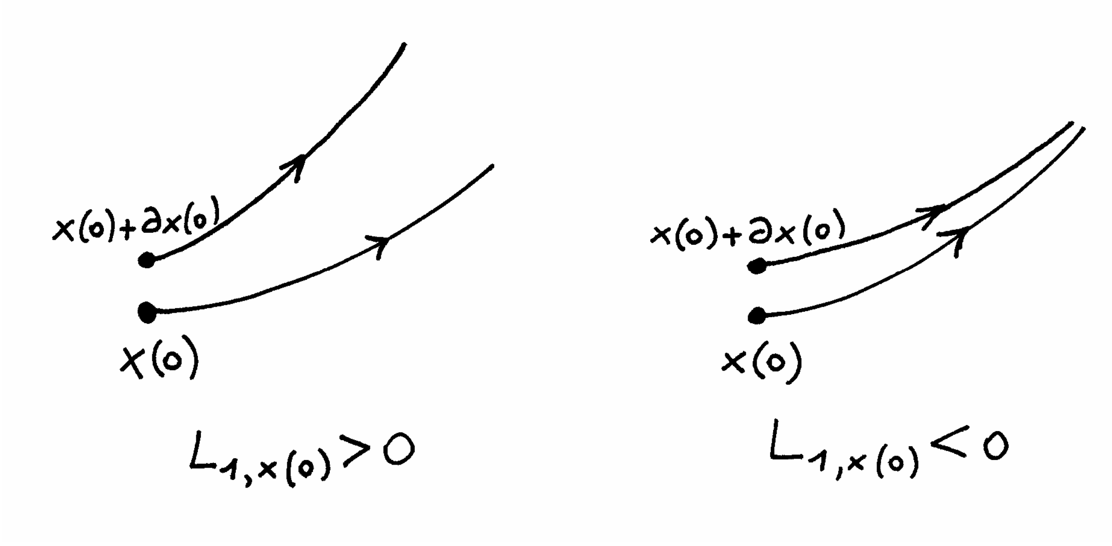


Remark: if we take a generic $\partial x(0)$ *(=not orthogonal to the maximum grow direction)*, asymptotically ($t \to \infty$) we have

$$\left|\partial x(t)\right| \to e^{L_{1,x(0)}\,t}\left|\partial x(0)\right|$$

because all the other terms (i.e. the other LEs) become negligible.

Therefore, the first (=maximum) LE specifies whether, on the average, nearby trajectories diverge ($L_{1,x(0)} > 0$) or converge ($L_{1,x(0)} < 0$).



CONTINUOUS-TIME SYSTEMS

Consider now

- the continuous-time system $\dot{x}(t) = f(x(t))$, with any order $n \geq 1$
- an initial state $x(0)$

A period-*T* map can be defined, which maps each state $x(0)$ into the state $x(T)$, i.e.

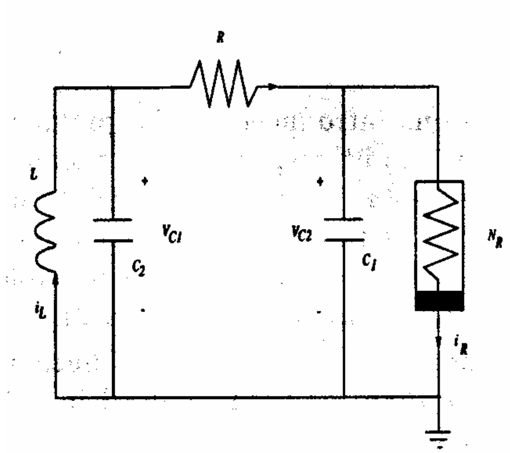
$$x((k+1)T) = F_T(x(kT)), \quad \text{with } T > 0 \text{ arbitrary}$$

It is a discrete-time system, whose LEs are $\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_n$.

Then the LEs of $\dot{x}(t) = f(x(t))$ are given by $L_i = \tilde{L}_i / T$, $i = 1,2,\ldots,n$.

Example: Chua circuit

It is a third-order electric circuit ($n=3$) with a nonlinear component:



$$C_1 \frac{dv_{C_1}}{dt} = \frac{1}{R}(v_{C_2} - v_{C_1}) - g(v_{C_1})$$

$$C_2 \frac{dv_{C_2}}{dt} = \frac{1}{R}(v_{C_1} - v_{C_2}) + i_L$$

$$L \frac{di_L}{dt} = -v_{C_2},$$

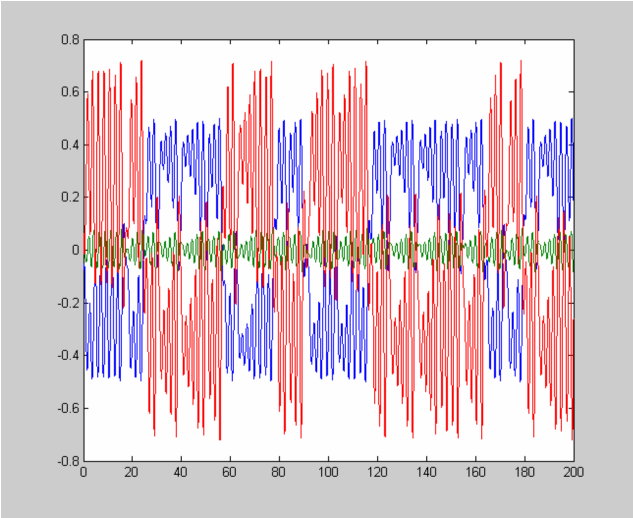
If the nonlinear component has a cubic voltage-current characteristic, the state equations become:

$$\dot{x} = \alpha(y - ax^3 - cx)$$

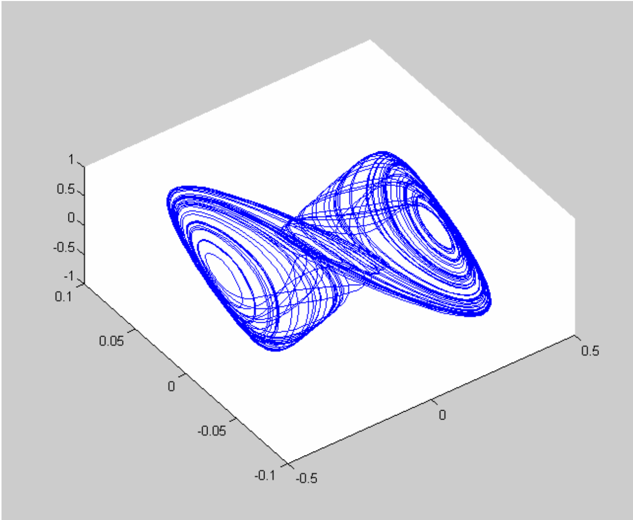
$$\dot{y} = x - y + z$$

$$\dot{z} = -\beta y$$

For suitable parameter values, the system has a nonperiodic trajectory:



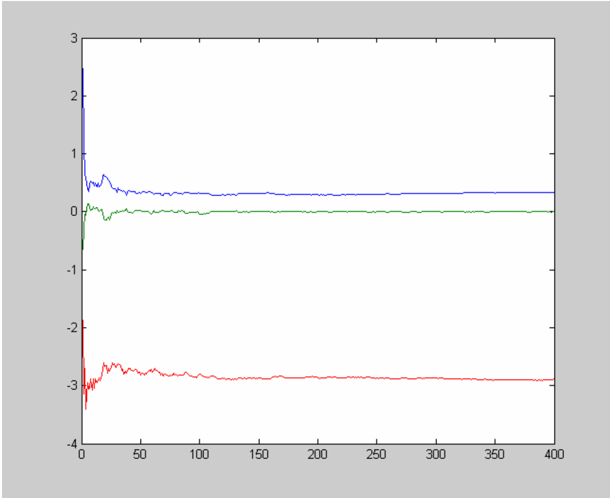
time series of x, y, z



state-space trajectory

The LEs are computed by an iterative algorithm.

$$L_1 > 0\,, \qquad L_2 \cong 0\,, \qquad L_3 < 0$$



CHAOTIC ATTRACTORS

- Attractors
- Classification of attractors: equilibria, limit cycles, tori, chaotic attractors
- LEs of attractors

ATTRACTORS

Consider a continuous- or discrete-time system

$$\dot{x} = f(x) \qquad \text{or} \qquad x(t+1) = f(x(t))$$

and denote by $x(t) = \Phi(t, x_0)$, $t \geq 0$, the orbit with initial state x_0.

Definition: A closed and bounded set $A \subset R^n$ is an attractor if

i) it is invariant

(i.e. $\Phi(t, A) \subset A$ for all $t \geq 0$)

(i.e. starting in A the trajectory remains in A forever)

ii) it is attractive

(i.e. there exists an open and invariant set $U \supset A$ such that $\Phi(t,U) \to A$ for $t \to +\infty$)

(i.e. starting in a neighborhood of A the trajectory will tend to A)

iii) it is minimal

(i.e. there is no proper subset of A satisfying conditions i) e ii))

Condition iii) can be replaced by alternative requirements (largely equivalent) that put in evidence important properties of attractors:

A is indecomposable (or topologically transitive):

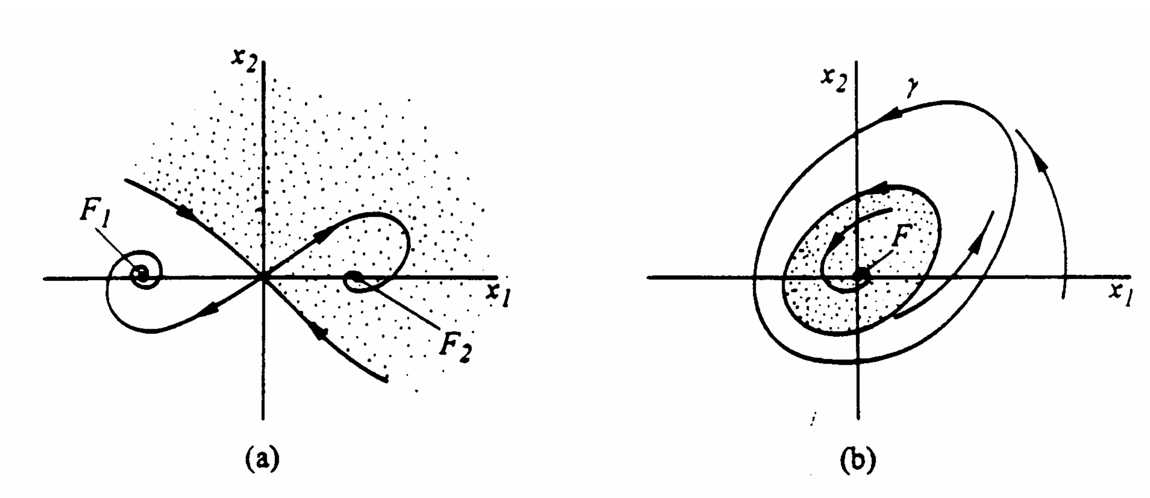
- For each pair of sets $X', X'' \subset A$ there exists $t \geq 0$ such that $\Phi(t, X') \cap X'' \neq 0$

A contains a dense orbit:

- There exists $x_0 \in A$ such that the set $\{\Phi(t, x_0) \mid t \geq 0\}$ is dense in A, i.e.
- Starting form a generic point of A, the trajectory will pass (in finite time) arbitrarily close to any point of A

The basin of attraction $B(A)$ is the set $B(A)=\{x \mid \Phi(t,x)\to A\}$, i.e. the set of initial states starting from which the trajectory tends to A.

Example: two state-space portraits with multiple attractors

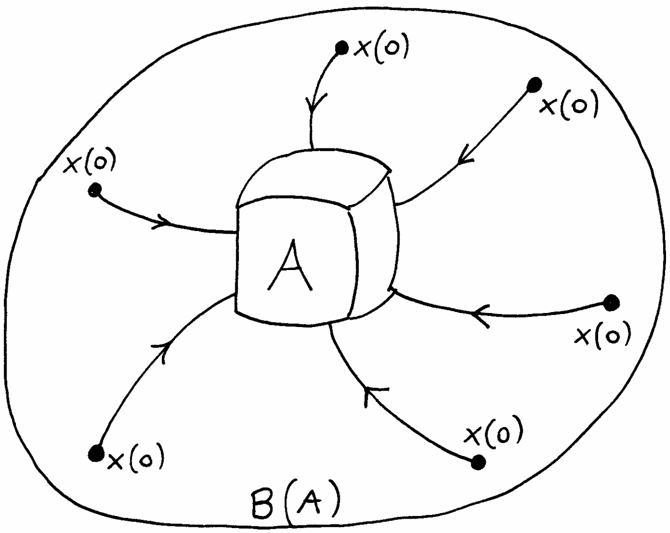


By definition, the LEs are related to a trajectory (=an initial state $x(0)$):

$$L_i = L_{i,x(0)} \quad , \quad i = 1,2,\ldots,n$$

In fact, given an attractor A, it can be proved that all trajectories starting from $x(0) \in B(A)$ (i.e. within the basin of attraction) have the same LEs.

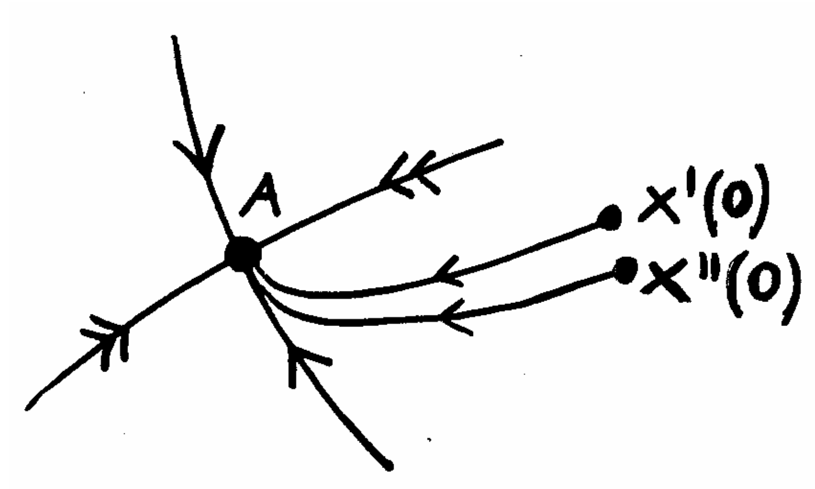
Thus the LEs give a characterization of the attractor A.



EQUILIBRIA

Since the equilibrium is an attractor, the distance between any two nearby trajectories decreases

$\Rightarrow$ the LEs are negative



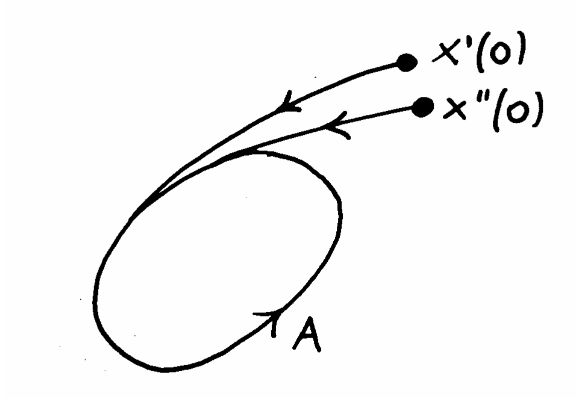
$$0 > L_1 \geq L_2 \geq \ldots \geq L_n$$

Remark: the same property holds for a cycle of a discrete-time system $x(t+1) = f(x(t))$, because:

period-T cycle of the map f = equilibrium of the map f^T

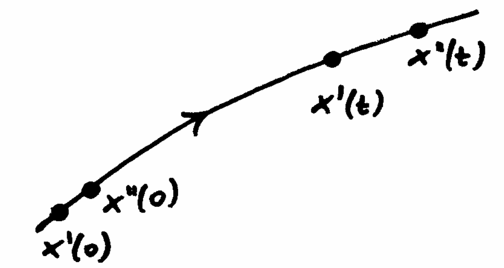
LIMIT CYCLES

Consider a limit cycle A of the time-continuous system $\dot{x} = f(x)$.



Any two nearby trajectories started within $B(A)$ tend to A, but their distance does not vanishes.

Indeed, the component of $\left(x'(t) - x''(t)\right)$ along the cycle remains unchanged (on the average):



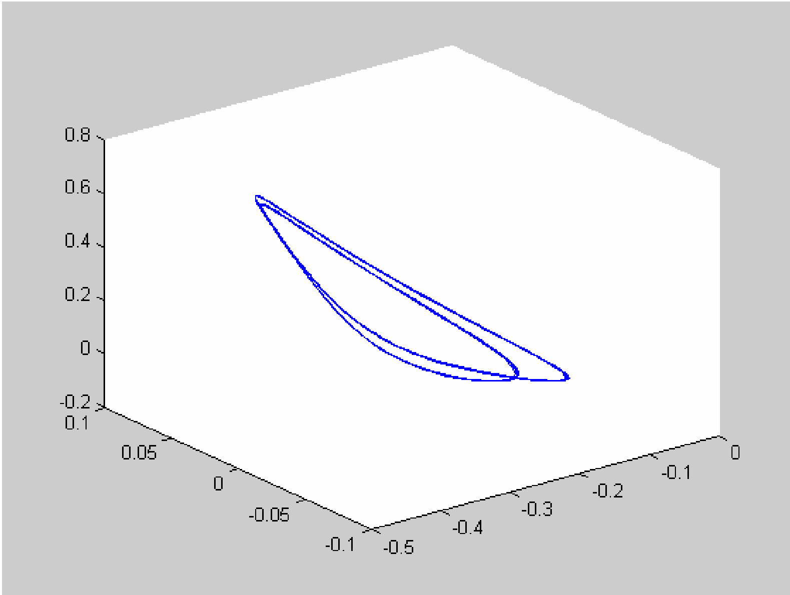
$\Rightarrow$ a LE is zero: $L_1 = 0$

Since the limit cycle is an attractor, the remaining LEs are negative:

$$0 = L_1 > L_2 \geq \ldots \geq L_n$$

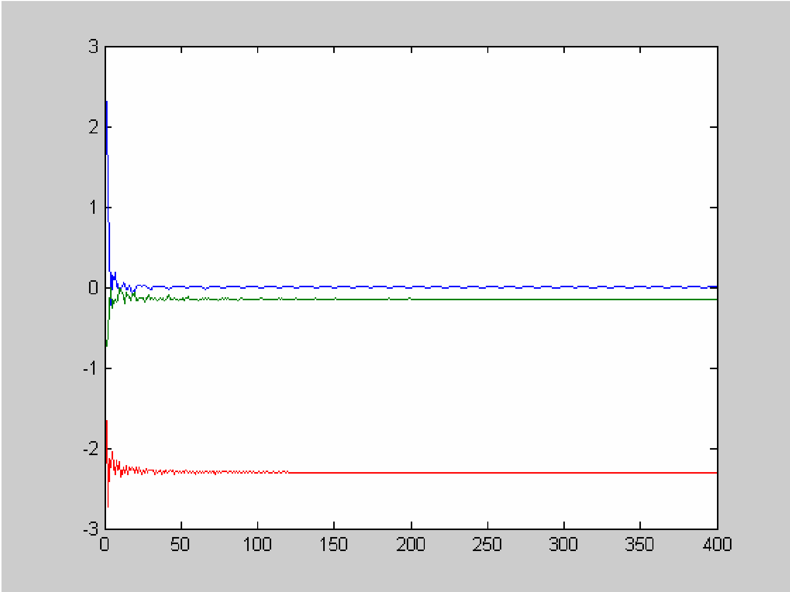
Example: Chua circuit

For suitable parameter values, the system has a periodic trajectory.



The LEs are computed by an iterative algorithm.

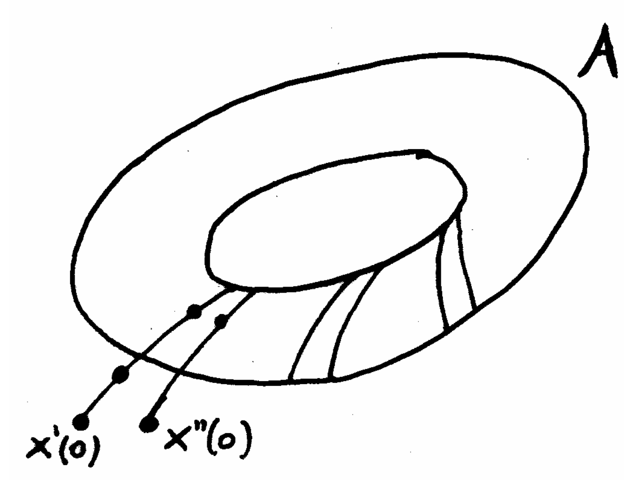
$$L_1 = 0.0008 \cong 0$$
$$L_2 = -0.1469$$
$$L_3 = -2.3077$$



TORI

Consider a torus A (generated by 2 frequencies) of the continuous-time system $\dot{x} = f(x)$.

As well as for a limit cycle, any two nearby trajectories started within $B(A)$ tend to A, but their distance does not vanishes.



However, now there are 2 components of $\left(x'(t) - x''(t)\right)$ that remain unchanged (on the average), so that

$$L_1 = L_2 = 0$$

Since the torus is an attractor, the remaining LEs are negative:

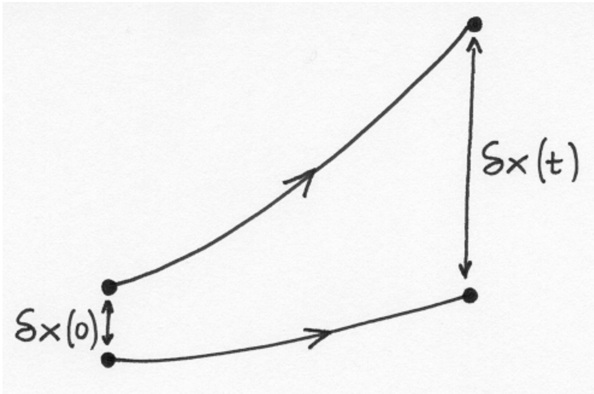
$$0 = L_1 = L_2 > L_3 \geq L_4 \geq \ldots \geq L_n$$

More generally, a k-torus (= k frequencies) has k LEs equal to zero.

CHAOS

Definition: A closed and bounded set $A \subset R^n$ is a chaotic attractor if

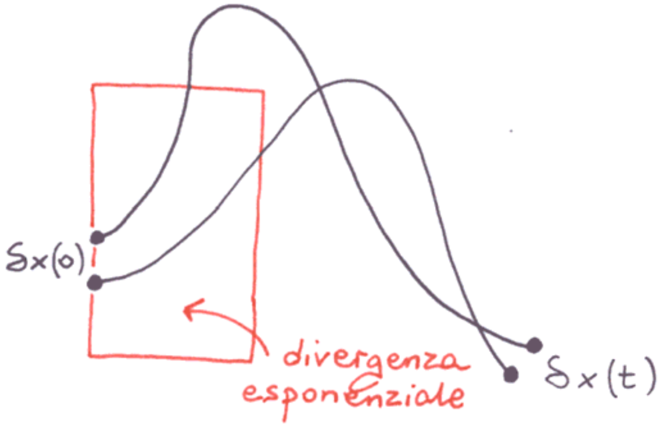
i) it is an attractor
ii) $L_1 > 0$



Therefore, in a chaotic attractor any two nearby trajectories exponentially diverge ("stretching").

However, if $\partial x(0)$ is finite (not infinitesimal), the grow of $\partial x(t)$ cannot continue indefinitely, because the attractor A is bounded.

The system nonlinearities will eventually take the two trajectories close again ("folding").

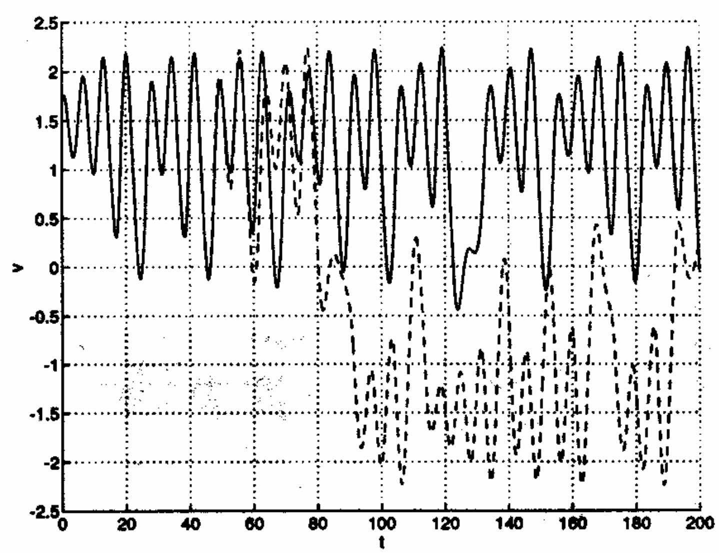


The "stretching" ($L_1 > 0$) gives rise to sensitive dependence on the initial conditions: arbitrarily close initial states ($|\partial x(0)| = \varepsilon > 0$) generate trajectories that become distant in finite time.

In other words, an arbitrarily small uncertainty on the initial state $x(0)$ makes $x(t)$ unpredictable in the medium/long term (the "butterfly effect").

Example: Chua circuit

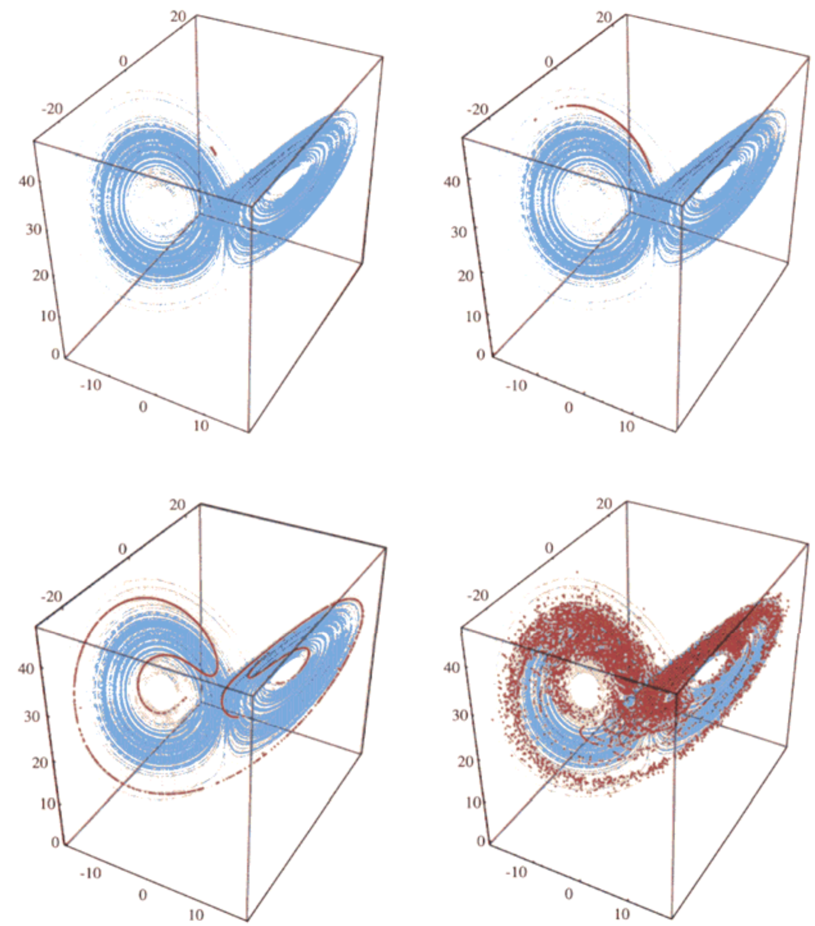
Two trajectories with initial distance $|\partial x(0)| = 10^{-3}$ separate after sometime, giving rise to different behaviors.



Example: Lorenz system

The evolution of a small ball of 10^4 initial states.

After sometime, the trajectories are practically uncorrelated.



Generically, given a chaotic attractor A:

- $k > 0$ LEs are positive because of the stretching (if $k > 1$ there are more than one directions of divergence: hyperchaos)

$$L_1 \geq L_2 \geq \ldots \geq L_k > 0$$

- for systems $\dot{x} = f(x)$, 1 LE is zero: the component of $\partial x(0)$ along the trajectory remains unchanged (on the average)

$$L_{k+1} = 0$$

- the remaining LEs are negative, because A is an attractor

$$0 > L_{k+2} \geq L_{k+3} \geq \ldots \geq L_n \quad , \quad \text{for } \dot{x} = f(x)$$

$$0 > L_{k+1} \geq L_{k+2} \geq \ldots \geq L_n \quad , \quad \text{for } x(t+1) = f(x(t))$$

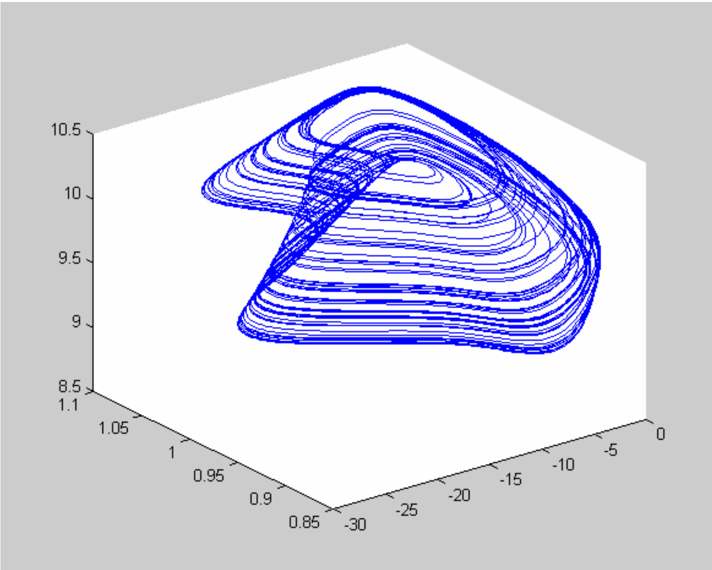
Example: CO_2 laser

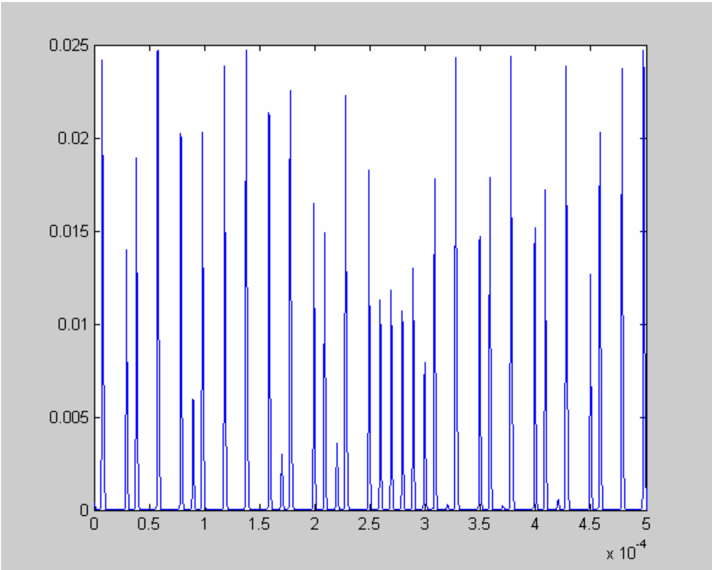
A simplified model:

$$\dot{x}_1 = a\left[x_2 - u - \left(1 + b\sin(ct)\right)\right]$$
$$\dot{x}_2 = -dx_2 + ex_3 - 2ax_2\exp(x_1) + e\left(f + g\right)$$
$$\dot{x}_3 = -hx_3 + lx_2$$

where $\exp(x_1)$ is proportional to the light intensity.

For suitable parameter values, the system has a nonperiodic behavior:



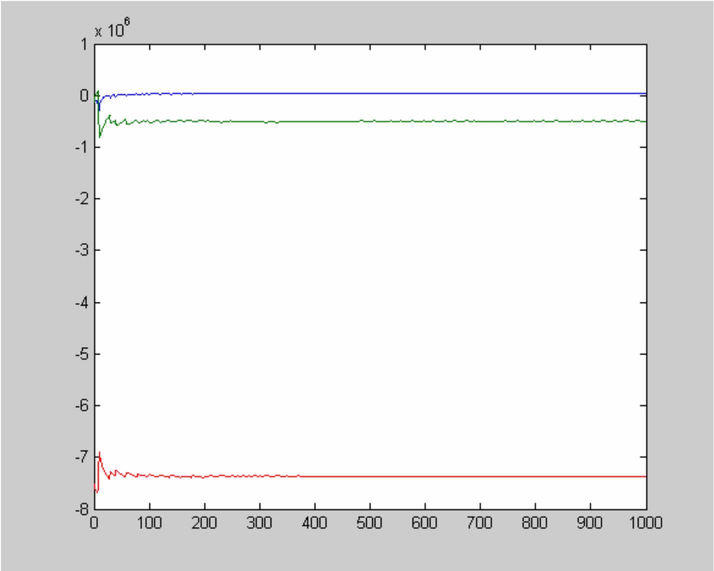


$\exp(x_1(t))$

The laser model is a periodic system $\dot{x} = f(t,x)$, with period $T = 2\pi / c$.

It is equivalent to the period-*T* map:

$$x((k+1)T) = F(x(kT)) \quad , \quad x \in R^3$$



The LEs are computed by an iterative algorithm.

$$L_1 = 0.361 \times 10^5 > 0$$

$$L_2 = -4.948 \times 10^5 < 0$$

$$L_3 = -73.675 \times 10^5 < 0$$

$L_1 > 0$ denotes that the behavior is chaotic.

Remark: in most cases, computing the first (maximum) LE L_1 allows one to classify the type of attractor.

	$\dot{x} = f(x)$	$x(t+1) = f(x(t))$
Equilibria	$L_1 < 0$	$L_1 < 0$
Limit cycles	$L_1 = 0 \ (L_2 < 0)$	$L_1 < 0$
Tori	$L_1 = 0 \ (L_2 = 0)$	$L_1 = 0$
Chaos	$L_1 > 0$	$L_1 > 0$

The computation of the first LE only can be done with efficient and numerically stable algorithms.

EXERCISES

1. (Numerical experiments on Lorenz system)

For each of the values of r given below, use a computer to explore the dynamics of the Lorenz system, assuming $\sigma = 10$ and $b = 8/3$. In each case, plot $x(t)$, $y(t)$, and x vs. z. You should investigate the consequences of choosing different initial conditions and lengths of integration. Also, in some cases you may want to ignore the transient behavior, and plot only the sustained long-term behavior.

$r = 10$; $r = 22$ (transient chaos); $r = 24.5$ (chaos and stable point co-exist); $r = 100$ (surprise); $r = 126.52$; $r = 400$

2. (Liapunov exponent of the logistic map)

For each of the values of r given below, compute all the equilibria of the logistic map $x(t+1) = rx(t)(1 - x(t))$ and study their stability. Then, use a computer to evaluate the Liapunov exponent, also investigating the consequences of choosing different initial conditions and lengths of integration.

$r = 0.5$ (trivial equilibrium); $r = 2$ (equilibrium); $r = 3.2$ (period-2 cycle); $r = 3.8$ (chaos); $r = 3.83$ (period-3 cycle)

3. (Ueda attractor)

Consider the system $\ddot{x} + k\dot{x} + x^3 = B\cos t$, with $k = 0.1$ and $B = 12$. Write the system equations in the usual form $\dot{z} = f(z)$ by defining a suitable two-dimensional vector z. Show numerically that the system has a chaotic attractor, and plot its Poincaré section.

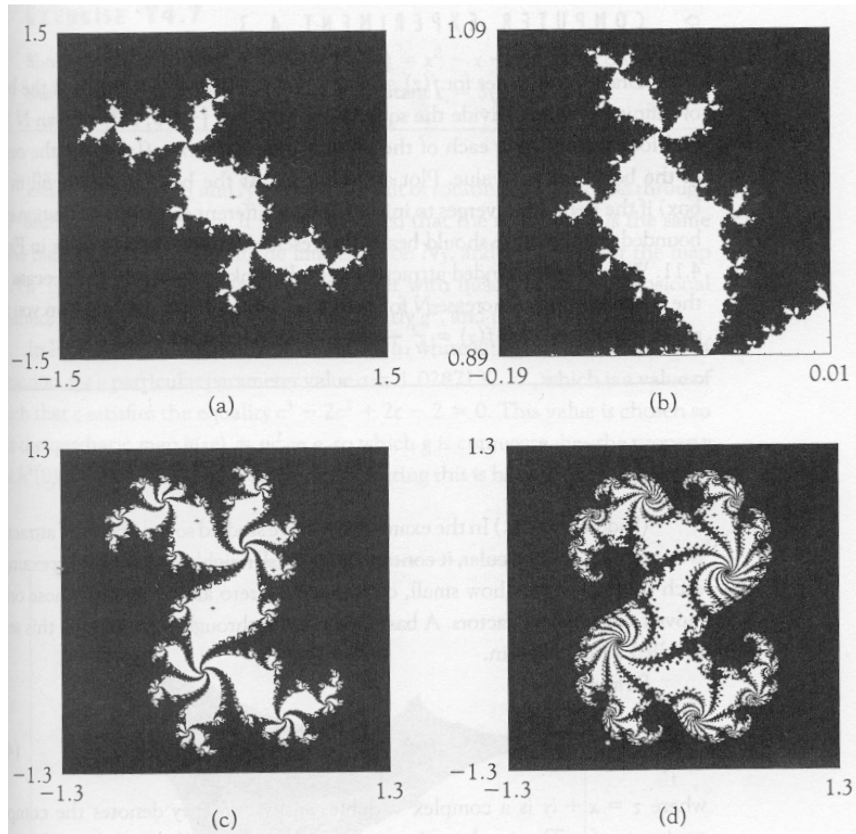
FRACTAL GEOMETRY

- Dimension of a set
- Elementary fractal sets
- Fractal dimensions
- Fractal geometry and dynamical systems

Typical features of a fractal set include

- complex structure at arbitrarily small scale
- self-similarity
- non integer dimension

Example: Julia sets ⇒



Many natural objects display such features:

coasts,
cabbages,
corals,
trees,
hydrological nets,
nervous system,
bronchial system,
Saturn rings,
...

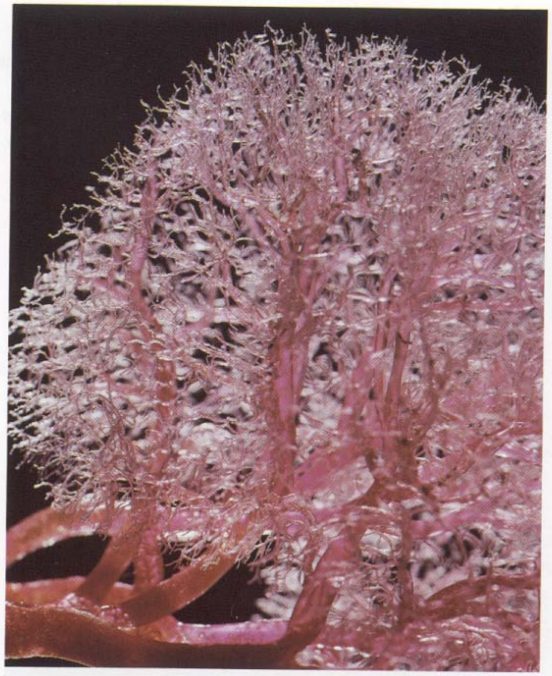


Plate 2: Cast of a child's kidney, venous and arterial system, © Manfred Kage, Institut für wissenschaftliche Fotografie.

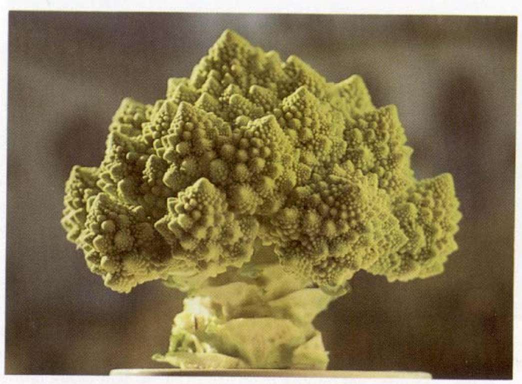


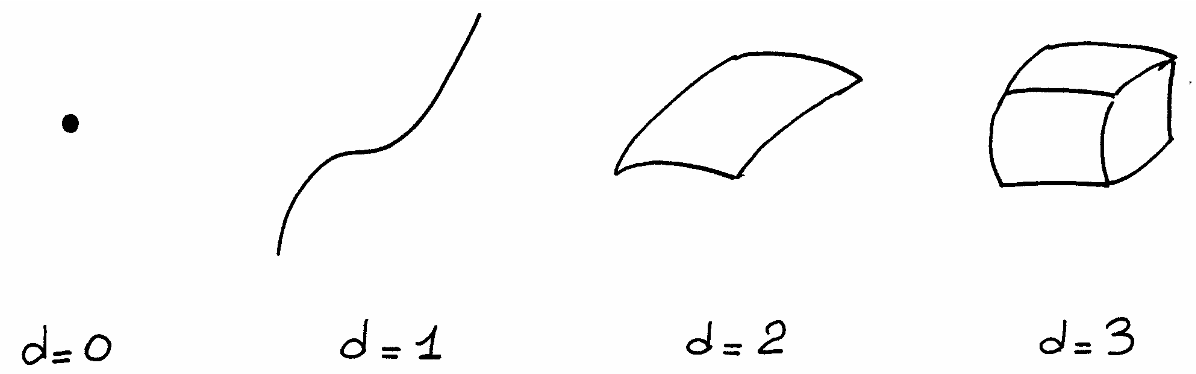
Plate 3: Broccoli Romanesco.

Plate 4: Wadi Hadramaut, Gemini IV image, © Dr. Vehrenberg KG.

DIMENSION OF A SET

Consider “simple” sets in R^n: a point, a smooth line, a smooth surface,….

Intuitively, we can state that the dimension is the number of coordinates needed to identify each point of the set.

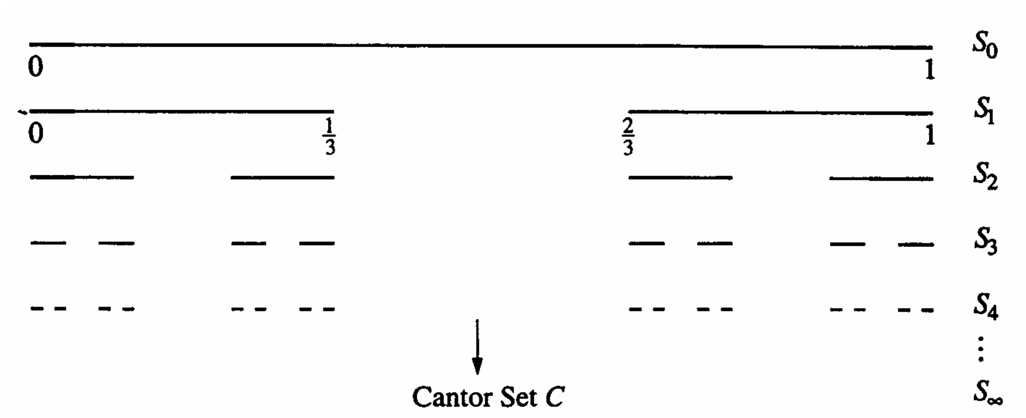


“Simple” sets have integer dimension, as well as the union of a countable number of them.

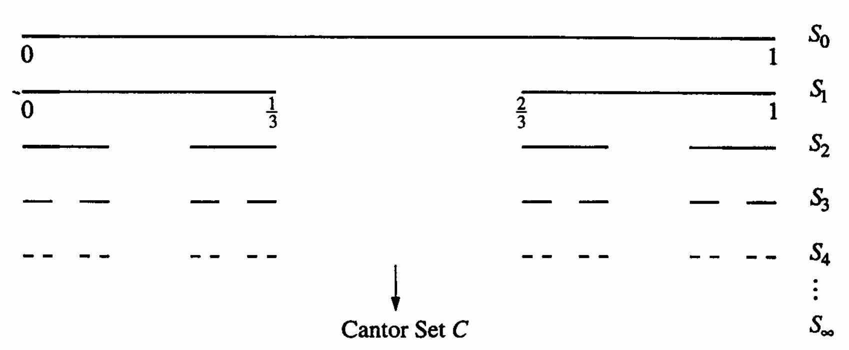
ELEMENTARY FRACTAL SETS

The (middle-third) Cantor set

Starting from $[0,1]$, at each step the "middle-third" of each segment is erased.



The Cantor set is the set $C = S_\infty$ which is obtained after infinite steps.



- C is self-similar: it contains copies of itself at any scale (e.g.: the part of C in $[0,1/3]$ is equal to the entire C, scaled by 3).

- C has zero length. Indeed, the length of the set S_{k+1} is $l_{k+1} = (2/3)l_k$ and thus tends to 0 as $k \to \infty$. Therefore the dimension of C is <1 ("it is less than a line…").

- C is an infinite set (=infinitely many points) and it is uncountable ("it is more than a point…").

We will learn that the dimension of C is non integer, between 0 and 1.

Generally speaking, we define a topological Cantor set as a set S such that:

- S is totally disconnected: S does not contains any connected subset, i.e. each point is "separated" from each other point.

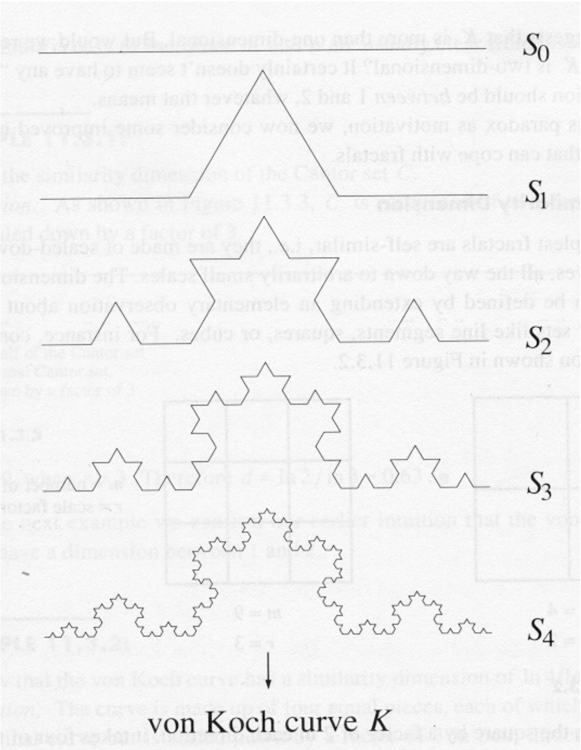
- S does not contains isolated points: in any arbitrarily small neighborhood of each point of S there are other points of S.

The "middle-third" Cantor set has the two above properties.

Typically, the chaotic attractors of discrete-time systems (and of Poincaré maps of continuous-time systems) are topological Cantor sets.

The von Koch curve

Starting from a segment S_0, at each step the "middle-third" of each sub-segment is erased and replaced by the other two sides of an equilateral triangle.



The von Koch curve is the set $K = S_\infty$ which is obtained after infinite steps.

Remark:

The von Koch curve K has infinite length.

Indeed, the length of S_{k+1} is $l_{k+1} = (4/3)l_k$ and thus tends to ∞ as $k \to \infty$. Therefore the dimension of K is >1 (“it is more than a line…”).

However, since K is a union of segments, its area is zero (“it is less than a surface…”).

We will learn that the dimension of K is non integer, between 1 and 2.

FRACTAL DIMENSIONS

Several criteria have been proposed to quantify the dimension of fractal sets ("fractal dimension"). We analyze three of them:

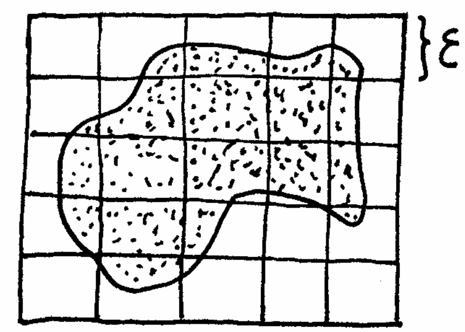
"Box-counting" dimension d_B

Correlation dimension d_C

Liapunov dimension d_L

"Box-counting" dimension

Consider a set $S \subset R^n$ contained in an n-dimensional "cube" H, and partition H in "boxes" whose side is ε.



The total number of boxes $T(\varepsilon)$ is proportional to $(1/\varepsilon)^n$.

Now denote by $N(\varepsilon)$ the number of boxes containing at least one point of S.

S has dimension d_B if, for small ε, $N(\varepsilon)$ obeys the power law

$$N(\varepsilon) = \gamma \left(\frac{1}{\varepsilon} \right)^{d_B} \qquad \text{or equivalently} \qquad \log N(\varepsilon) = \log \gamma + d_B \log(1/\varepsilon)$$

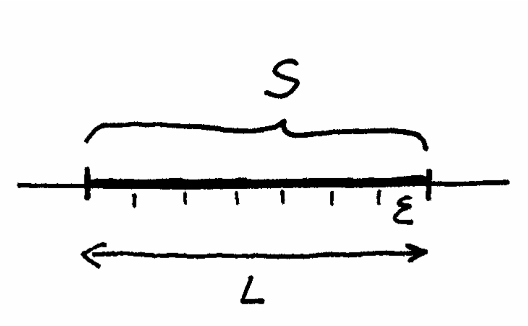
Letting $\varepsilon \to 0$, we have the definition of "box-counting" dimension

$$d_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

Example: "simple" sets

$n = 1$

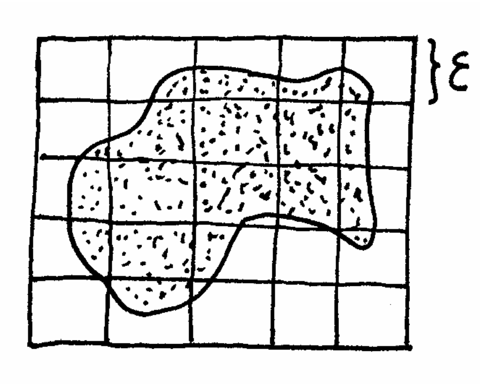
A segment with length L is covered by $N(\varepsilon) = L / \varepsilon$ boxes.



Thus $d_B = 1$.

$n = 2$

A surface with area A is covered, for $\varepsilon \to 0$, by $N(\varepsilon) \to A / \varepsilon^2$ boxes.



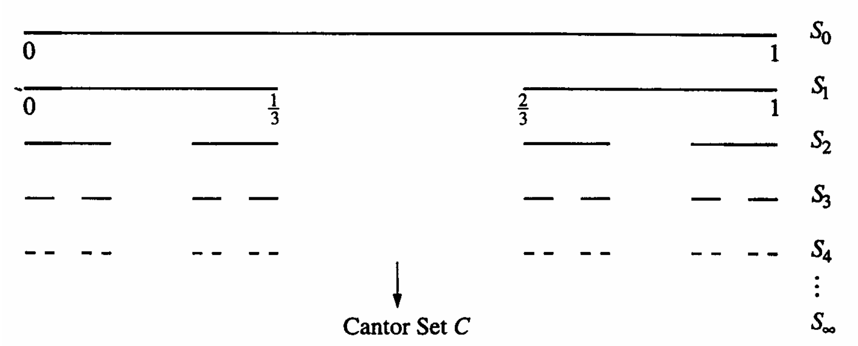
Thus $d_B = 2$.

Example: "middle-third" Cantor set

The set S_k is covered by $N(\varepsilon) = 2^k$ intervals each of length $\varepsilon = (1/3)^k$.

$$d_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\log 2^k}{\log 3^k} = \lim_{k \to \infty} \frac{k \log 2}{k \log 3} = \frac{\log 2}{\log 3} \cong 0.63093$$

The dimension of the Cantor set C is non-integer (= fractal).

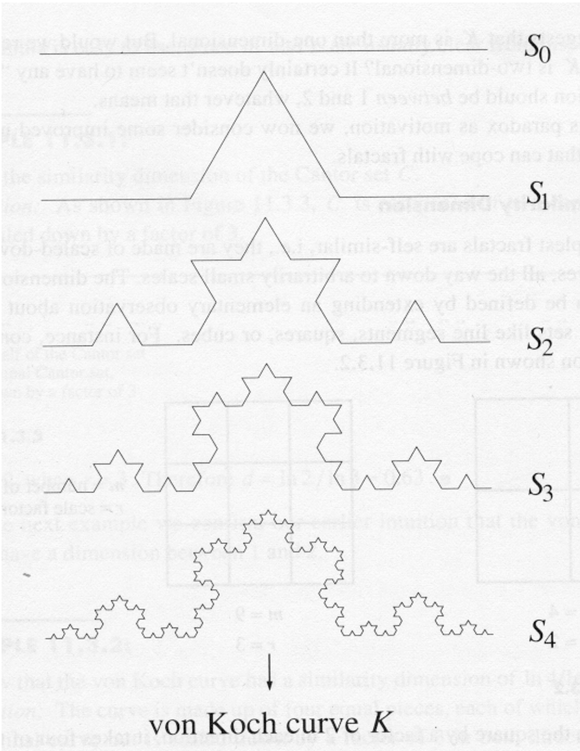


Example: von Koch curve

The set S_k is covered by $N(\varepsilon) = 4^k$ intervals each of length $\varepsilon = (1/3)^k$.

$$d_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\log 4^k}{\log 3^k} = \lim_{k \to \infty} \frac{k \log 4}{k \log 3} = \frac{\log 4}{\log 3} \cong 1.2618$$

The dimension of the von Koch curve is non integer (= fractal).

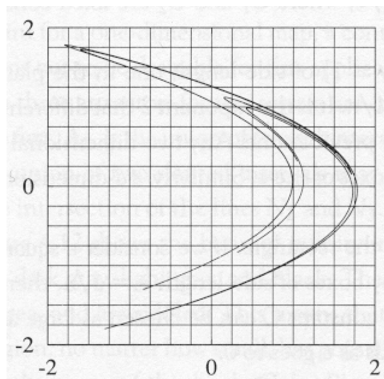


In most cases, the dimension has to be computed numerically.

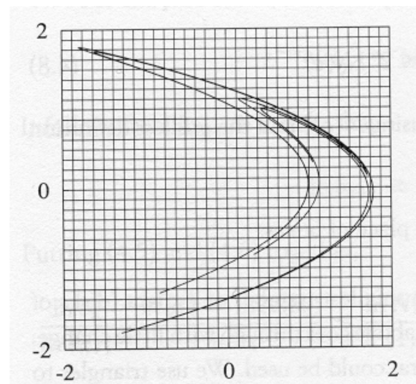
Example: the attractor of the Henon map

$$x(t+1) = y(t) + 1 - ax(t)^2 \qquad y(t+1) = bx(t)$$

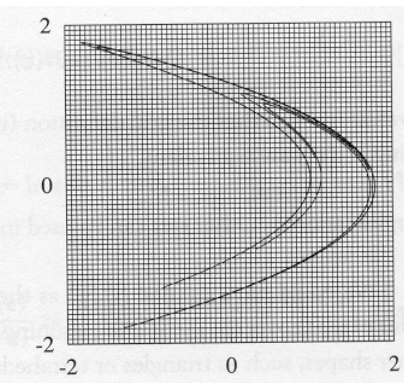
$N(\varepsilon)$ is computed for decreasing values of ε, and plotted with respect to $(1/\varepsilon)$.



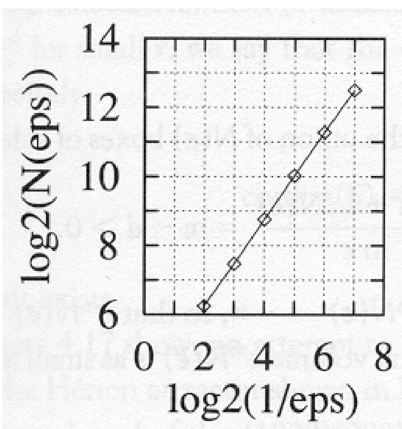
$\varepsilon = 1/4$



$\varepsilon = 1/8$



$\varepsilon = 1/16$



Since $\log N(\varepsilon) = \log\gamma + d_B \log(1/\varepsilon)$, the slope (on log scales) is the estimate of d_B.

In this case $d_B \cong 1.27$.

Correlation dimension

It is related to a set $S = \{x(0), x(1), \cdots\}$, which is typically the trajectory of a discrete-time system $x(t+1) = f(x(t))$.

Given $r > 0$, the correlation function $C(r)$ is defined as the number of pairs of points of S (w.r.t. the total number of pairs) whose distance is less than r:

$$C(r) = \lim_{t \to \infty} \frac{\#\,\mathrm{pairs}\big(x(i), x(j)\big)\ \mathrm{s.t.}\ \|x(i) - x(j)\| < r}{\#\,\mathrm{pairs}\big(x(i), x(j)\big)}$$

S has dimension d_C if, for small r, $C(r)$ obeys the power law

$$C(r) = \gamma\, r^{d_C} \qquad \text{or equivalently} \qquad \log C(r) = \log \gamma + d_C \log r$$

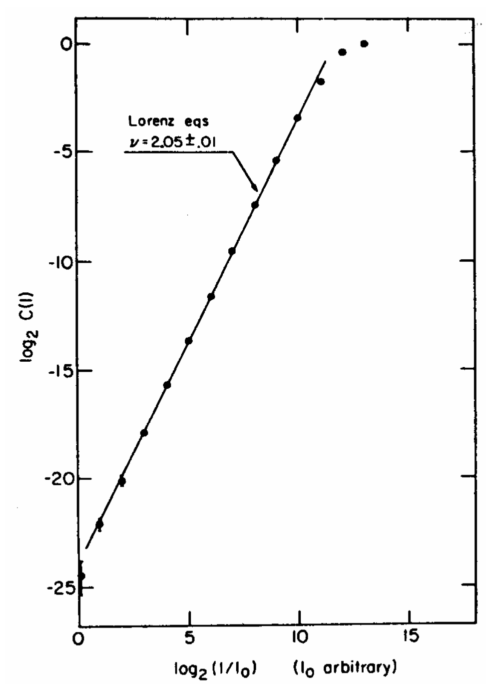
Letting $r \to 0$, we have the definition of correlation dimension

$$d_C = \lim_{r \to 0} \frac{\log C(r)}{\log r}$$

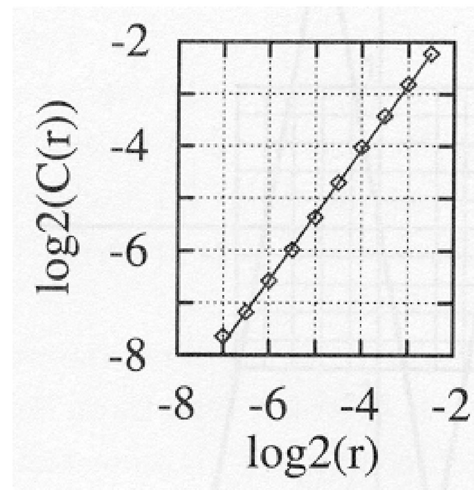
[Remark:](#) given a trajectory, $C(r)$ is numerically computed for several r and then $C(r)$ is plotted w.r.t. r.

Since $\log C(r) = \log\gamma + d_C \log r$, the slope (on log scales) is the estimate of d_C.

Example: Lorenz system and Henon map in chaotic regime



$d_C \cong 2.05$



$d_C \cong 1.23$

Remark: as the system order n increases, the computation of the correlation dimension d_C becomes more convenient than that of the "box-counting" d_B.

As a matter of fact, the number of boxes needed for computing d_B grows exponentially with n.

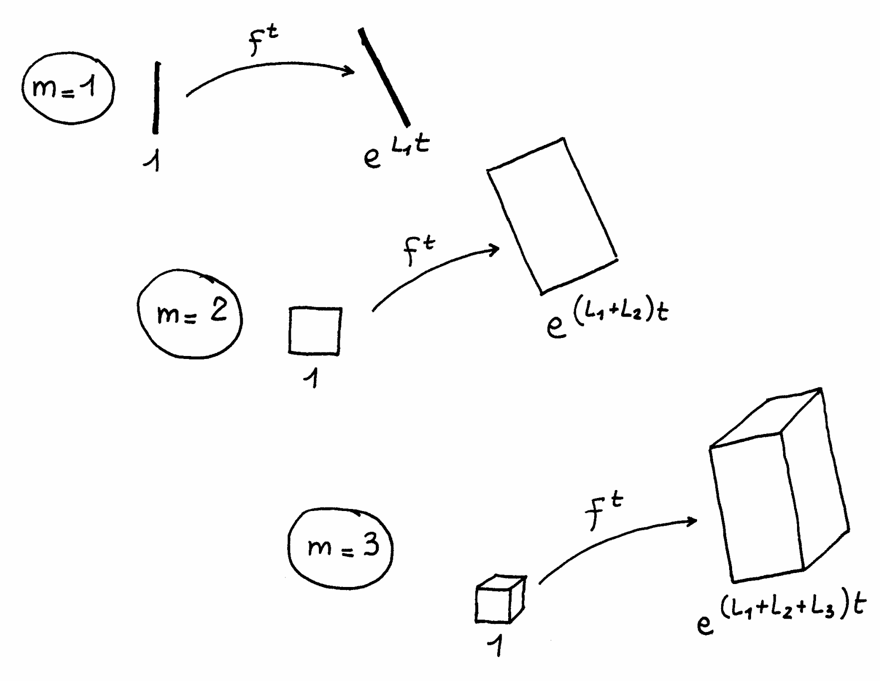
It can be shown that, for any set S, the following inequality holds:

$$d_B \geq d_C$$

Liapunov dimension

It is related to an attractor A, whose Liapunov exponents are

$$L_1 \geq L_2 \geq \cdots \geq L_n$$



It can be shown that, for all $m \leq n$

$$\exp(S_m) = \exp(L_1 + L_2 + \cdots + L_m)$$

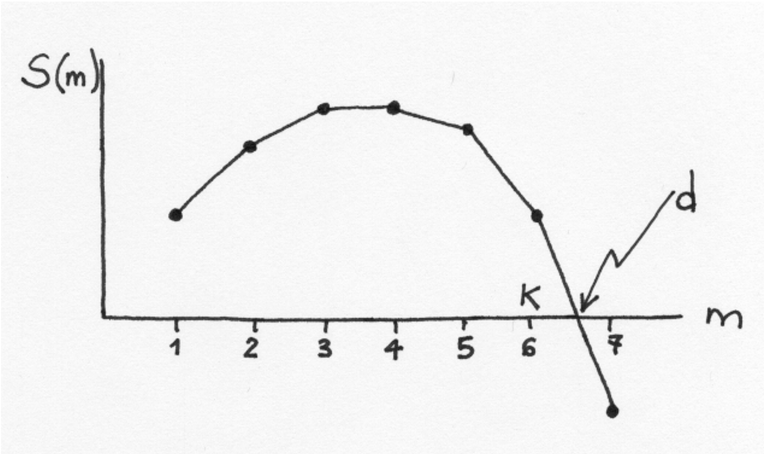
is the average expansion rate (if >1) or contraction rate (if <1) of the m-dimensional volumes along the trajectory.

Typical (dissipative) systems have $S_n < 0$.

If A is a chaotic attractor then $S_1 = L_1 > 0$. Thus S_m (as a function of m) is typically shaped as in the figure.

Note that

- $S_k > 0$, i.e. k-dim volumes expand
- $S_{k+1} < 0$, i.e. $(k+1)$-dim volumes contract



There exists a non-integer value d_L ($k < d_L < k+1$) such that d_L-dim volumes remain unchanged.

$\Rightarrow$ The attractor A has dimension d_L.

Kaplan-Yorke formula gives an estimate of d_L by linear interpolation:

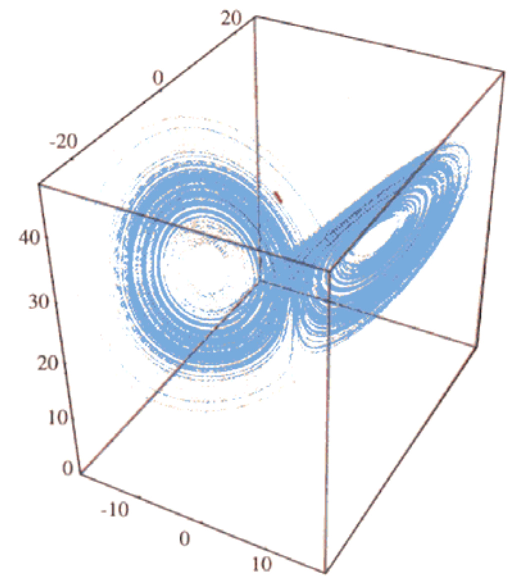
$$d_L = k + \frac{S_k}{|L_{k+1}|} \qquad \text{, where } k = \max\{m \mid S_m > 0\}$$

Example: Henon map ($n = 2$)

Computing the Liapunov exponents gives $L_1 = 0.39$, $L_2 = -1.59$. Thus $k = 1$ and

$$d_L = 1 + 0.39/1.59 = 1.25$$

Example: Lorenz system ($n = 3$)



$L_1 = 0.905$, $L_2 = 0$, $L_3 = -14.57$.

Thus $k = 2$ and

$$d_L = 2 + 0.905/14.57 = 2.062$$

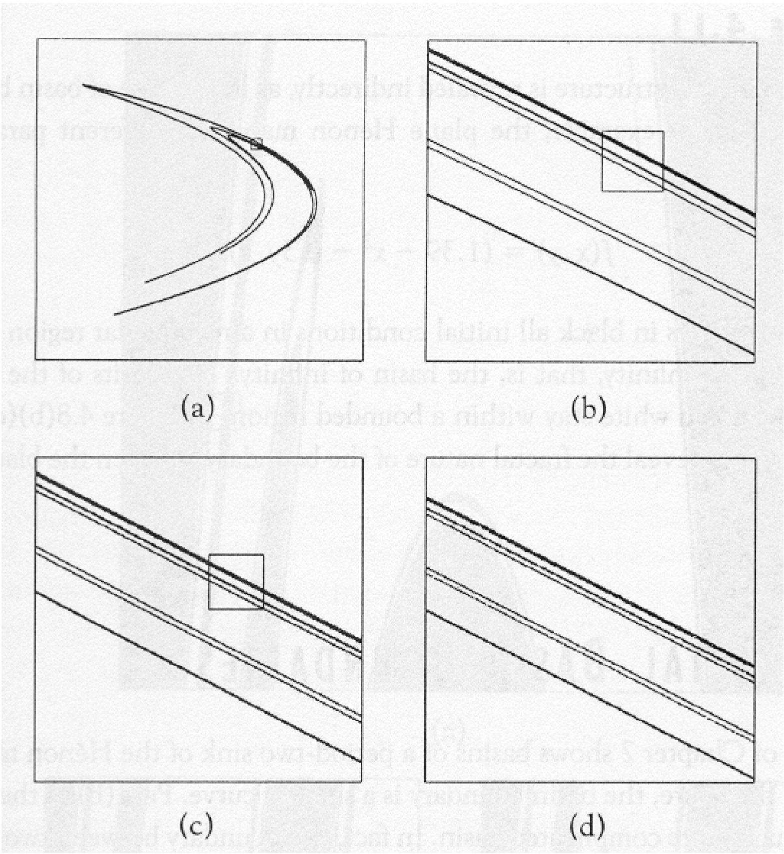
FRACTAL GEOMETRY AND DYNAMICAL SYSTEMS

When studying nonlinear dynamical systems, fractal geometries are found in many circumstances.

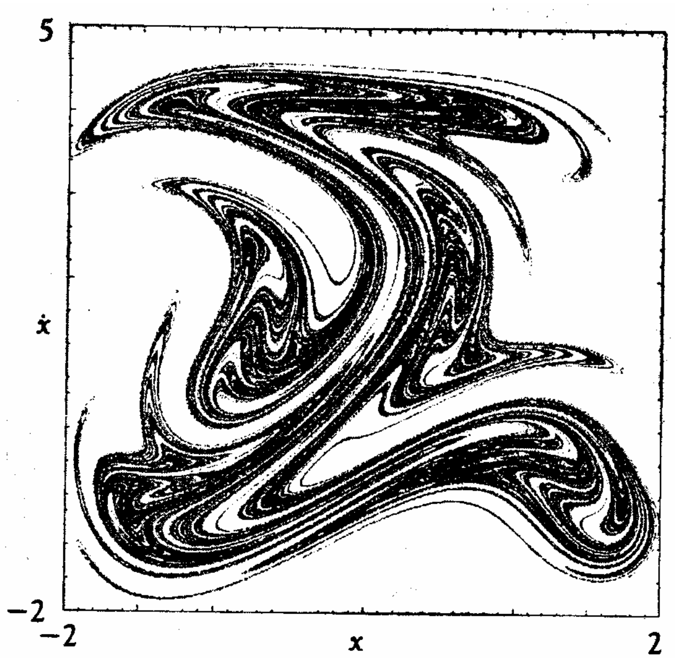
- Typically (but not always…), chaotic attractors are fractal sets.
- A basin of attraction (of an equilibrium, of a limit cycle, of a chaotic attractor, even of the infinity…) can have fractal boundary.
- In the space of the system parameters, the regions where the system has a given qualitative behavior can have fractal boundary.

Chaotic attractors

Example: Henon system
(discrete-time, $n = 2$)

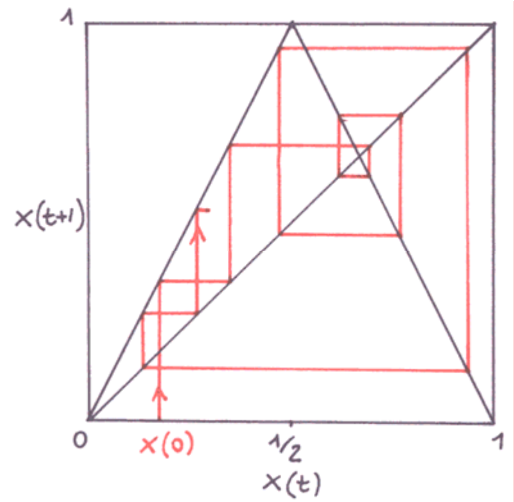


Example: periodically forced mechanical system
(continuous-time, periodic, $n = 2$)



Typically, a chaotic attractor has fractal dimension. But there are exceptions:

Tent map: $L_1 = \ln 2 > 0$ (chaos) but $x(t)$ densely covers the interval $[0,1]$ (thus $d = 1$, integer).



Logistic map: at $r = r_\infty = 3.5699456...$ (the border of chaos), we have $d_C = 0.5$ (fractal) but $L_1 = 0$.

chaotic attractor $\neq$ fractal attractor ("strange")

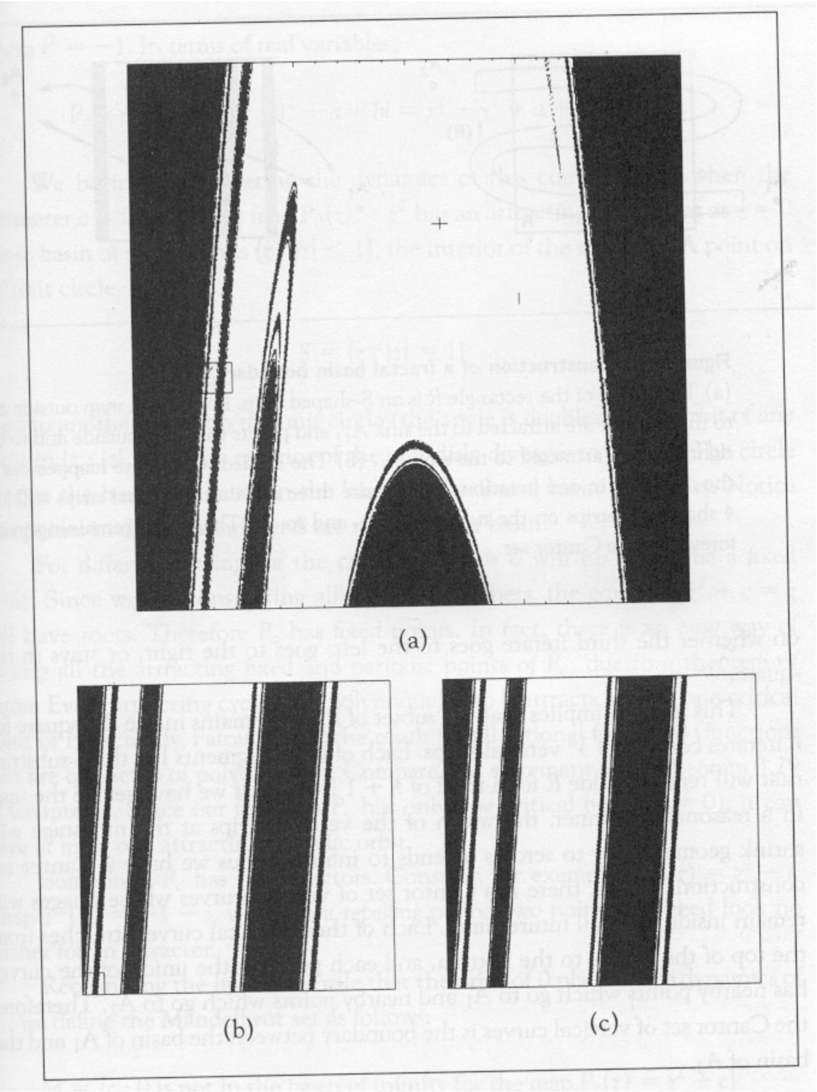
($L_1 > 0$) (d non integer)

Basins of attraction

Example: Henon map

The white region is the basin of attraction of a period-2 cycle.

The set has the same structure at arbitrarily small scale (self-similarity).



Parametric portraits

Example: Mandelbrot set

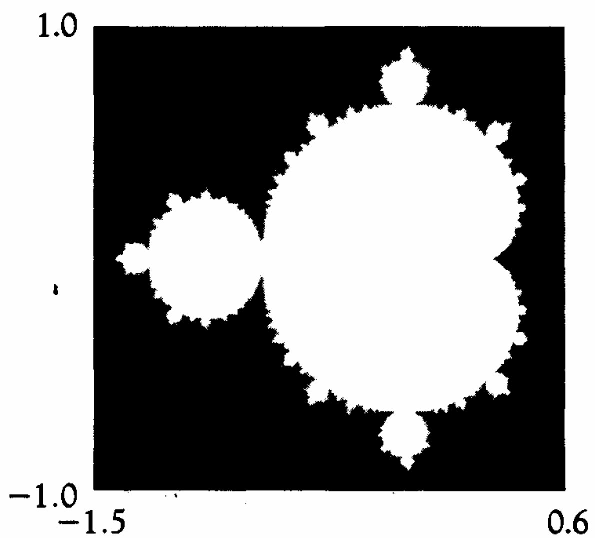
The map

$$z(t+1)=z(t)^2+c$$

where z and c are complex, is equivalent to a real-valued 2-nd order system (2 state variables, $x=\mathrm{Re}(z)$ and $y=\mathrm{Im}(z)$, and 2 parameters, $a=\mathrm{Re}(c)$ and $b=\mathrm{Im}(c)$).

In the complex plane of c, the Mandelbrot set M (the white set in the figure) is the set of parameter values such that the trajectory started at $z=0$ remains bounded.

The boundary of M is a fractal set.

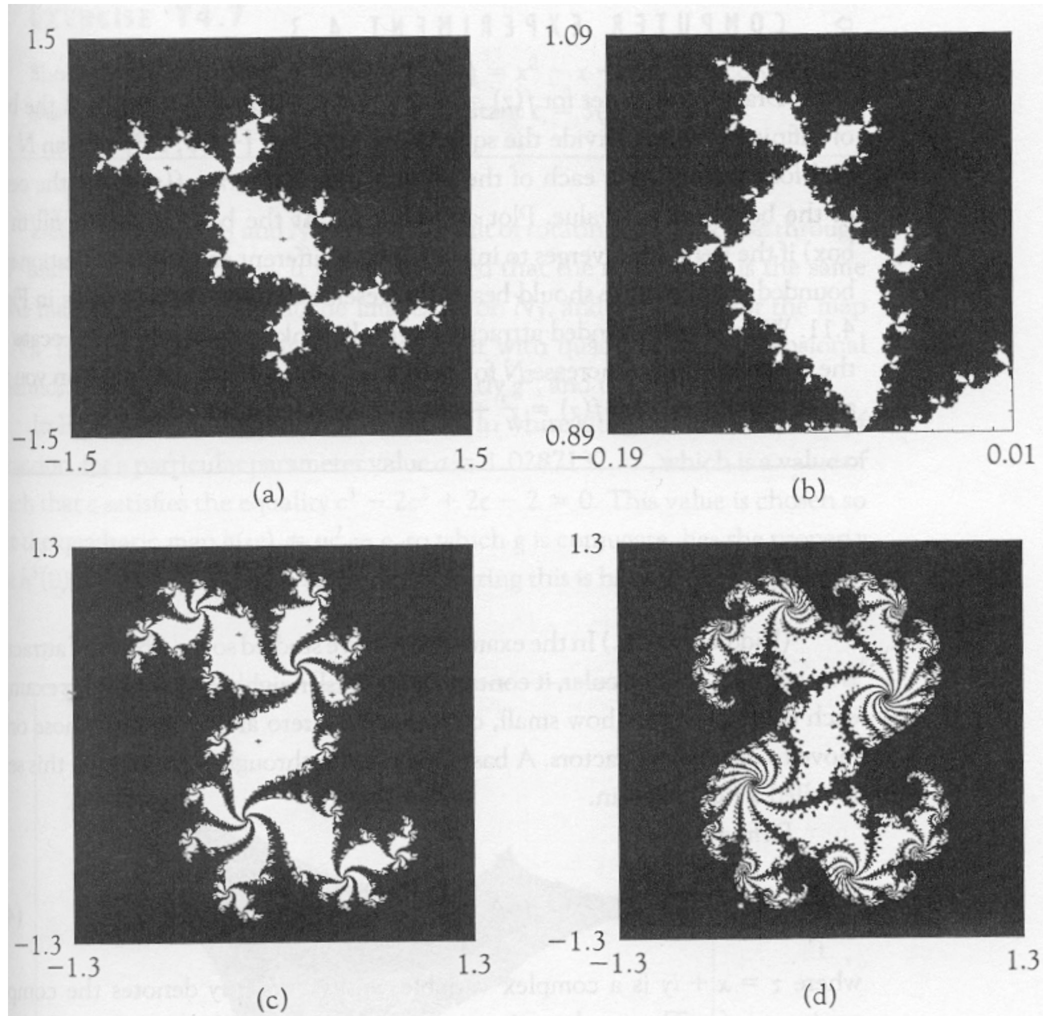


Given $c \in M$ (a point of the Mandelbrot set), there exists a set B of initial states $z(0)$ giving rise to bounded trajectories (B is non-empty, as it contains at least $z(0) = 0$).

The boundary of B is called Julia set, and it is a fractal set.

In the figure, the initial states $z(0)$ giving rise to bounded trajectories are depicted in white.

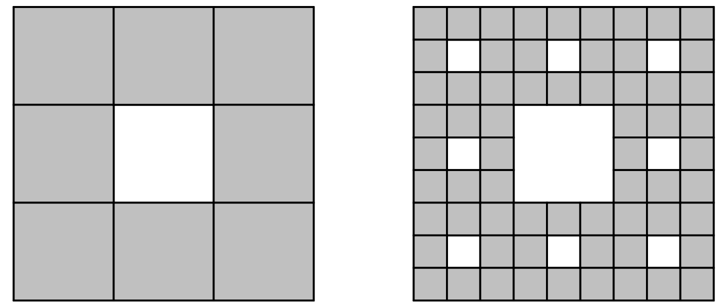
(a) $c = -0.17 + 0.78i$
(b) zooming into figure (a)
(c) $c = 0.38 + 0.32i$
(d) $c = 0.32 + 0.043i$



EXERCISES

1. (Sierpinski carpet)

Consider the process shown in the figure. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on. The figure shows the first two stages.



S_1 S_2

a) Sketch the next stage S_3.

b) Show that the limiting fractal, known as Sierpinski carpet, has zero area.

c) Find the box-counting dimension.

2. (Fractal attractor)

Consider the forced pendulum $\ddot{\theta} + b\dot{\theta} + \sin\theta = F\cos t$, with $b = 0.22$, $F = 2.7$.

a) Starting from any reasonable initial condition, use numerical integration to compute $\dot{\theta}(t)$. Show that the time series has an erratic appearance, and interpret it in terms of the pendulum's motion.

b) Plot the Poincaré section by strobing the system whenever $t = 2\pi k$, where k is an integer.

c) Zoom in on part of the strange attractor found in (b). Enlarge a region that reveals the fractal features of the attractor.

3. (Fractal basin boundary)

Consider again the pendulum of exercise 2, but now let $b = 0.2$, $F = 2$.

a) Show that there are two stable fixed points in the Poincaré section. Describe the corresponding motion of the pendulum in each case.

b) Compute the basins for each fixed point. Use a reasonably fine grid of initial conditions, and then integrate from each one until the trajectory has settled down to one of the fixed points (establish a criterion for the convergence). Show that the boundary between the basins looks like a fractal.