Deterministic chaos

1. How it looks like (lec.5.1 pp.2-17)
   - in time-series
   - in power spectra
   - in state portraits
   - on Poincaré sections
   - self-similarity
   - sensitivity to initial conditions

2. How it is defined (this page)
   - 2.1: non-equilibrium, non-periodic, non quasiperiodic deterministic dynamics (dynamics on a <u>strange attractor</u>)
   - 2.2: deterministic dynamics on a <u>fractal attractor</u>
   - 2.3: deterministic dynamics sensitive to initial conditions (first <u>Lyapunov exponent</u> $L_1 > 0$) (dynamics on a <u>chaotic attractor</u>)

   Note: there are also strange/fractal/chaotic saddles and repellors

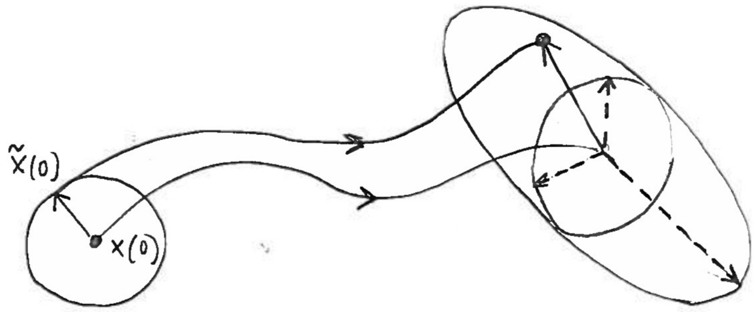
3. Lyapunov exponents: $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_n$. A generalization of the concept of eigenvalues when linearizing around a non-stationary non-periodic trajectory.
   (addendum + some pages in lec.5.1 pp. 18-46)

4. Fractal geometry
   (some pages in lec 5.2 pp.2-30)

5. The (typical) dynamical structure of a chaotic attractor

3. Lyapunov exponents

- Given a reference initial condition $x(0)$ and an ε-sphere of perturbed initial conditions $\tilde{x}(0)$, $\|\delta x(0)\| = \|\tilde{x}(0) - x(0)\| = \varepsilon > 0$, the linearized dynamics around the reference trajectory $x(t)$ transform the sphere at time 0 into an ellipsoid at time t.



c.t. $\begin{cases} \dot{x}(t) = f(x(t)) \\ \dot{\delta x}(t) = J(x(t)) \cdot \delta x(t) \end{cases}$

d.t. $\begin{cases} x(t+1) = f(x(t)) \\ \delta x(t+1) = J(x(t)) \cdot \delta x(t) \end{cases}$

Note: this result holds for any $\varepsilon > 0$, but the effect of the h.o.t. is small only if ε is small

- Denoting with $r_i(t)$ the lengths of the symmetry semi-axes of the ellipsoid, $r_1(t) \geq r_2(t) \geq \dots \geq r_n(t)$, the average rate of change of r_i is given by

$$\rho_i = \lim_{t \to +\infty} \left(\frac{r_i(t)}{\varepsilon} \right)^{1/t} \quad \text{(geometric average)}$$

- The i-th Lyapunov exponent is defined as

$$L_i = \ln \rho_i = \lim_{t \to +\infty} \frac{1}{t} \ln r_i(t)$$

It is the average exponential rate of expansion (if >0) / contraction (if <0) of the i-th semi-axis, i.e. $r_i(t) \approx \exp(L_i t)$

- Discrete-time: $r_i(t) = \sigma_i[H_{x(0)}(t)]$, $H_{x(0)}(t) = J(x(t-1)) \cdot \dots \cdot J(x(1)) \cdot J(x(0))$

  a sign-equivalent approximation $r_i(t) \simeq |\lambda_i[H_{x(0)}(t)]|$

  (useful to link LEs to equilibria's eigenvalues and cycles' multipliers)

  $n=1$: both formulas give $r_1(t) = \prod_{k=0}^{t-1} |f'(x(k))|$, so that $L_1 = \lim_{t \to +\infty} \frac{1}{t} \sum_{k=0}^{t-1} |f'(x(k))|$

- Continuous-time: $L_i = \frac{L_i^{(T)}}{T}$, where $L_i^{(T)}$ are the LEs of the T-stroboscopic map.

- Examples: see lec. 5-1 pp. 22-24, 28, 29

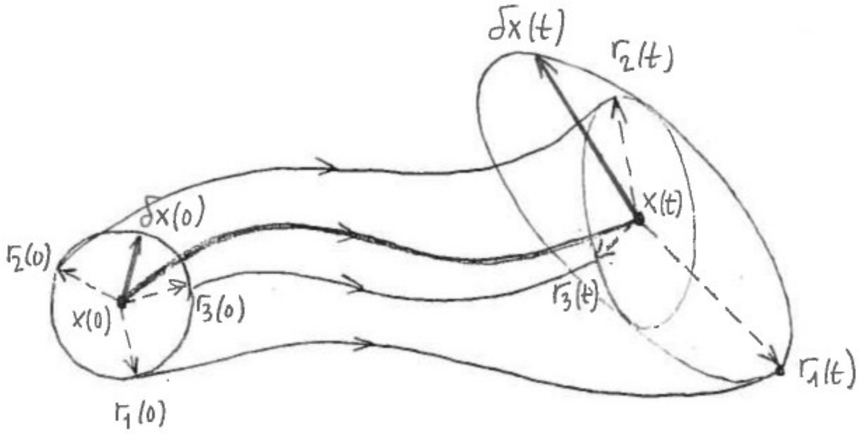
- Interpretations (see next. p.)

- Computation: see next lab and Matlab LET (Lyapunov Exponents Toolbox).

- LEs and attractors: see lec. 5.1 pp. 34-45

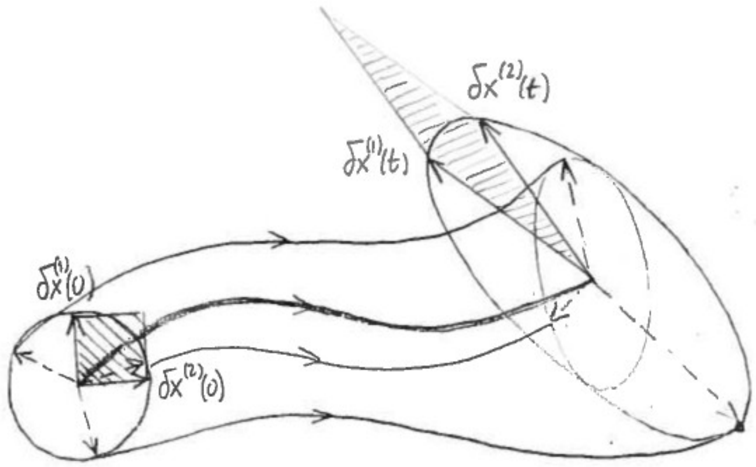
▶ Expansion / contraction of k-dimensional volumes

Note 1: for any $t > 0$ there is an orthogonal base $r_i(0)$ that is transformed into the ellipsoid's semi-axes at time t.



Note 2: generically, a perturbation $\delta x(0)$ has nonzero components w.r.t. all axes $r_i(0)$.

$k=1$: at time t, the component of $\delta x(t)$ along $r_1(t)$ dominates the components along $r_i(t)$, $i \geq 2$, so that, on average and up to 1-st-order, the length of $\delta x(t)$ grows/decays as $\|\delta x(0)\| \exp(L_1 t)$



$k=2$: at time t, the components of $\delta x^{(1)}(t)$ and $\delta x^{(2)}(t)$ along $r_1(t)$ and $r_2(t)$ dominate the others, so that, ..., the area of the parallelogram $(\delta x^{(1)}(t), \delta x^{(2)}(t))$ grows/decays as $\|\delta x^{(1)}(0)\| \|\delta x^{(2)}(0)\| \exp((L_1+L_2)t)$

$k>2$: the k-dimensional (hyper-)volume of the parallelotope $(\delta x^{(1)}(t), \dots, \delta x^{(k)}(t))$ ... grows/decays as $\|\delta x^{(1)}(0)\| \cdots \|\delta x^{(k)}(0)\| \exp((L_1 + \dots + L_k)t)$.

▶ An experimental interpretation of the LEs

L_1: take two nearby initial points (one reference, one perturbed) and follow their distance... For how long? For long to "feel" L_1 (it's the average growth-rate); for short if $L_1 > 0$ (L_1 is defined through linearization). There are two ways to get around this problem:

- theoretical way: take the two points very close (theoretically infinitesimally close), so that they will remain close for long (theoretically forever) even if $L_1 > 0$.
- practical way: repeat the experiment many times; each time compute the short-time exponential growth-rate; then average (arithmetically) the results.

$L_1 + L_2$: take three nearby initial points (one ref., two perturbed) and follow the area of their parallelogram.

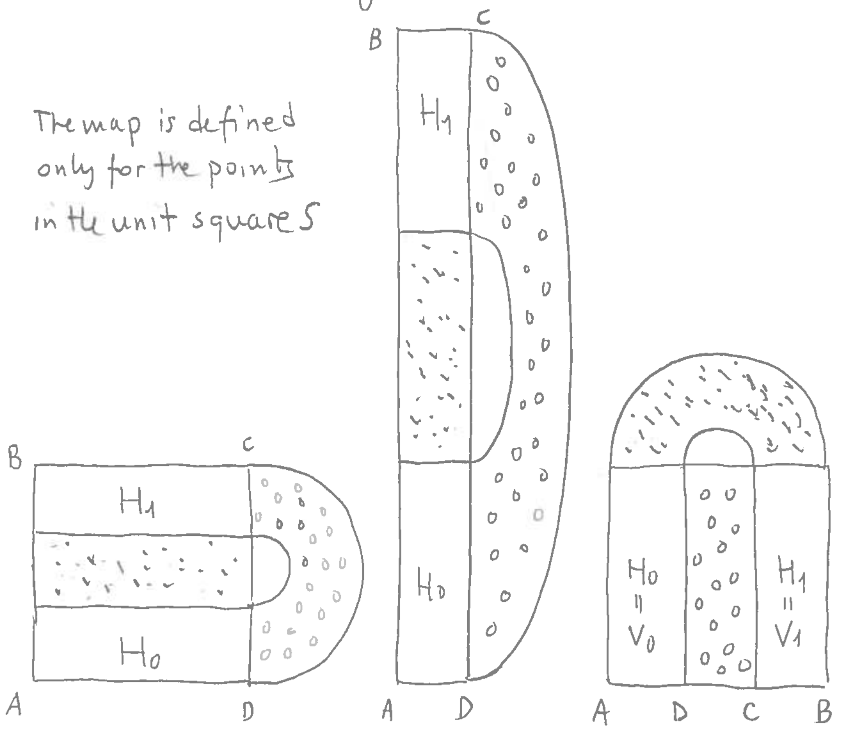
$L_1 + \dots + L_k$, $k > 2$: take $k+1$ nearby initial points (one ref., k perturbed) and follow the k-dim (hyper-)volume of their parallelotope.

5. The (typical) dynamical structure of a chaotic attractor A

- A (typically) contains a countable infinity of saddle cycles, the union of which densely fills A. A is thus the closure of the ensemble of cycles, that forms the "backbone" of A.
- The trajectory starting from a generic initial condition $x(0) \in A$ densely visit A.
- The trajectories starting from two close initial conditions $x'(0)$ and $x''(0)$ on A diverge exponentially (at an average rate $L_1 > 0$), but come back arbitrarily close ($\|x'(t) - x''(t)\| < \varepsilon$) at a later time.
- The dynamics on A is the result of a mechanism of stretching, responsible of local divergence ($L_1 > 0$), and of a mechanism of folding, keeping A bounded.
- The typical example (in discrete time, $n = 2$, reversible) of "stretching & folding" is the Smale horseshoe map.

"a thorough understanding of the Smale horseshoe map is absolutely essential for understanding what is meant by the term chaos" (Wiggins)

The map is defined only for the points in the unit square S



- There is an invariant set Λ of points that remain in S for ever (forward and backward in time)
- Λ contains a finite number of cycles for any period (even period 1 : equilibria)
- all cycles (and equilibria) are saddles (eigs = $\frac{1}{3}$, 3)
- Λ contains aperiodic orbits densely visiting Λ
- $L_1 = \log 3 > 0$! (sensitivity to initial conditions)
- If the map is extended outside S and a domain $\Omega \supset S$ is attractive, then Ω contains a chaotic attractor.

A dynamics equivalent to a Smale horseshoe is typically seen in discrete-time chaotic systems or on the Poincaré (or stroboscopic) maps of continuous-time chaotic systems.

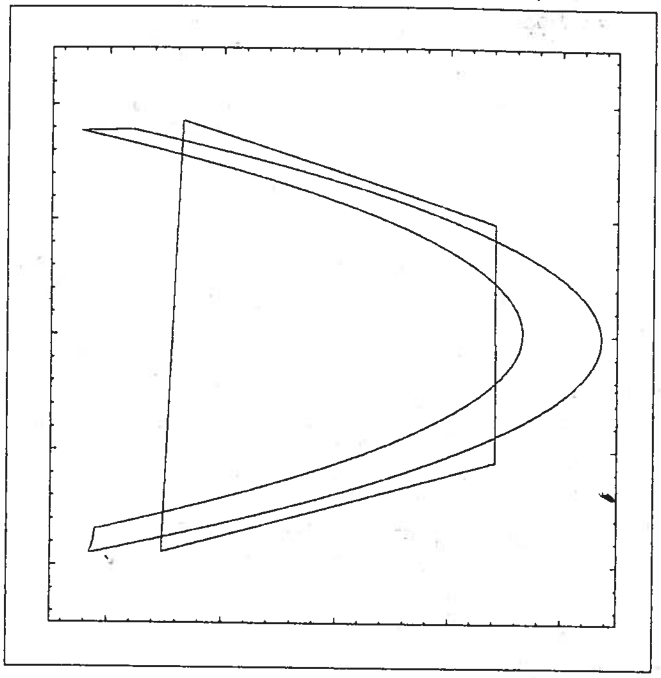


Figure 5.14 A horseshoe in the Hénon map.
A quadrilateral and its horseshoe-shaped image are shown. Parameter values are $a = 4.0$ and $b = -0.3$.

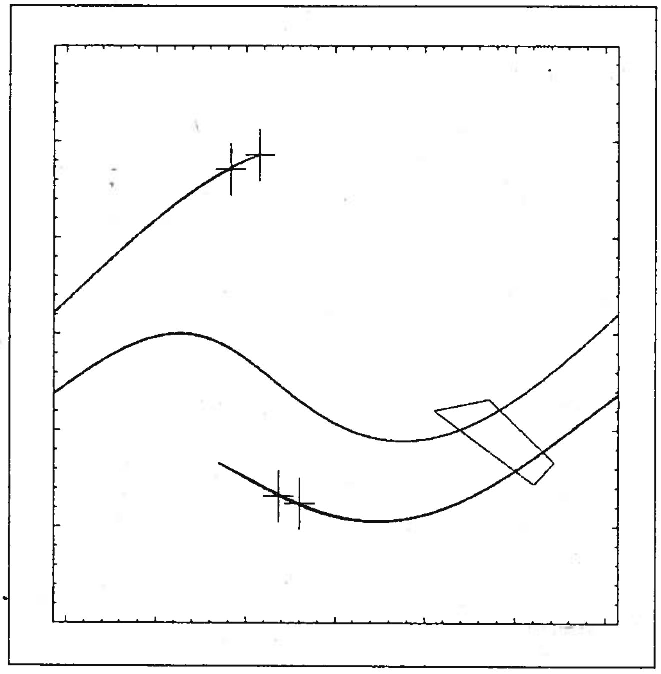


Figure 5.19 Horseshoe in the forced damped pendulum.
The rectangular-shaped region is shown along with its first image under the time-2π map. The image is stretched across the original shape, and is so thin that it looks like a curve, but it does have width. The crosses show the image of the corner points of the domain rectangle.