

Ex 1: Two runners

$$\begin{aligned} \dot{\rho}_1 &= \rho_1(1-\rho_1^2) & , & & \dot{\rho}_2 &= \rho_2(1-\rho_2^2) & , & & x_1 &= \rho_1 \cos \theta_1 & , & x_2 &= \rho_1 \sin \theta_1 \\ \dot{\theta}_1 &= \omega_1 & , & & \dot{\theta}_2 &= \omega_2 & , & & x_3 &= \rho_2 \cos \theta_2 & , & x_4 &= \rho_2 \sin \theta_2 \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_1 - \omega_1 x_2 - (x_1^2 + x_2^2) x_1 \\ \dot{x}_2 &= \omega_1 x_1 + x_2 - (x_1^2 + x_2^2) x_2 \\ \dot{x}_3 &= x_3 - \omega_2 x_4 - (x_3^2 + x_4^2) x_3 \\ \dot{x}_4 &= \omega_2 x_3 + x_4 - (x_3^2 + x_4^2) x_4 \end{aligned} \quad , \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$y = Ax_2 + Bx_4$$

Notes:

- ▶ The invariant torus is the 2-dimensional manifold $\{x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\}$
- ▶ ω_1/ω_2 rational \Rightarrow there are infinitely many stable (but not asymptotically stable) cycles on the torus. Fixing e.g. $x_1(0) = 1, x_2(0) = 0$, two different initial conditions with $x_3^2(0) + x_4^2(0) = 1$ belong to two different cycles.
- ▶ ω_1/ω_2 irrational \Rightarrow the dynamics on the torus is quasiperiodic. Each trajectory on the torus is aperiodic and densely fill the torus. Trajectories starting from close initial conditions remain close for all $t > 0$.
- ▶ The torus is K -dimensional, $K \geq 2$, if there are K incommensurable frequencies (f_i/f_j irrational for any $i \neq j$, $i, j = 1, \dots, K$) in the system.

Ex 2: Two friends running

$$\begin{aligned} \dot{x}_1 &= \dot{\rho}_1 \cos \theta_1 - \rho_1 \sin \theta_1 \dot{\theta}_1 = \rho_1(1-\rho_1^2) \cos \theta_1 - \rho_1 \sin \theta_1 (\omega_1 + K_1 \overbrace{\sin(\theta_2 - \theta_1)}^{\sin \theta_2 \cos \theta_1 - \sin \theta_1 \cos \theta_2}) \\ &= x_1 - \omega_1 x_2 - (x_1^2 + x_2^2) x_1 - \frac{K_1 x_2}{\sqrt{x_1^2 + x_2^2} \sqrt{x_3^2 + x_4^2}} (x_4 x_1 - x_2 x_3) \end{aligned}$$

Notes:

- ▶ Similarly for $\dot{x}_2, \dot{x}_3, \dot{x}_4$
- ▶ $|\omega_1 - \omega_2| < K_1 + K_2 \Rightarrow$ there are two cycles on the torus (one asympt. stable, the other unstable). The phase difference on the stable cycle is constant (phase locking)

Quasiperiodicity in discrete time

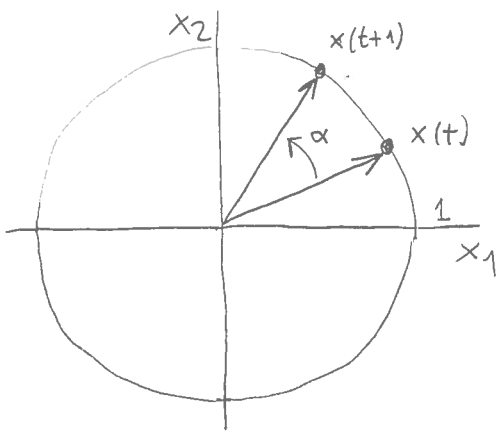
$$y(t) = A \sin(\alpha t)$$

$$\alpha/\pi = \begin{cases} \text{rational} & (\text{non generic}) \\ \text{irrational} & (\text{generic}) \end{cases}$$

$$\alpha/\pi = \frac{p}{q} \Rightarrow 2q\alpha = p2\pi \Rightarrow y \text{ is periodic with period } T = 2q$$

$$\alpha/\pi \text{ irrational} \Rightarrow y \text{ is } \underline{\text{quasiperiodic}}$$

EX: Rotation



$$x(t+1) = R x(t) \quad (\text{linear system})$$

+ h.o.t. to make the unit circle an attractive invariant curve*

$$R = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad (\text{rotation matrix})$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y(t) = A x_2(t)$$

Notes:

- ▶ The invariant "torus" is the unit circle $x_1^2 + x_2^2 = 1$. It is called "torus" by analogy with the continuous-time case. It is an invariant curve.
- ▶ α/π rational \Rightarrow there are infinitely many stable (not asymp. stable) cycles. E.g. two different initial conditions on the unit circle belong to two different cycles.
- ▶ α/π irrational \Rightarrow the dynamics on the torus is quasiperiodic (same comments as in continuous time).
- ▶ The torus is k -dimensional, $k \geq 1$, if there are k incommensurable rotations angles (α_i/α_j irrational for any $i \neq j, i, j = 1, \dots, k$) in the system.

* for example

$$\rho(t+1) = \rho(t) + \frac{1-\rho(t)}{2} = \frac{1+\rho(t)}{2} \Rightarrow \bar{\rho} = 1 \text{ is globally stable}$$

$$\vartheta(t+1) = \vartheta(t) + \alpha$$

$$x_1(t+1) = \rho(t+1) \cos \vartheta(t+1) = \frac{1}{2}(1+\rho(t))(\cos \vartheta(t) \cos \alpha - \sin \vartheta(t) \sin \alpha)$$

$$= \frac{1}{2}(x_1(t) \cos \alpha - x_2(t) \sin \alpha) + \frac{1}{2\sqrt{x_1^2(t) + x_2^2(t)}}(x_1(t) \cos \alpha - x_2(t) \sin \alpha)$$

$$x_2(t+1) = \dots = \frac{1}{2}(x_1(t) \sin \alpha + x_2(t) \cos \alpha) \left(1 + \frac{1}{\sqrt{x_1^2(t) + x_2^2(t)}} \right)$$