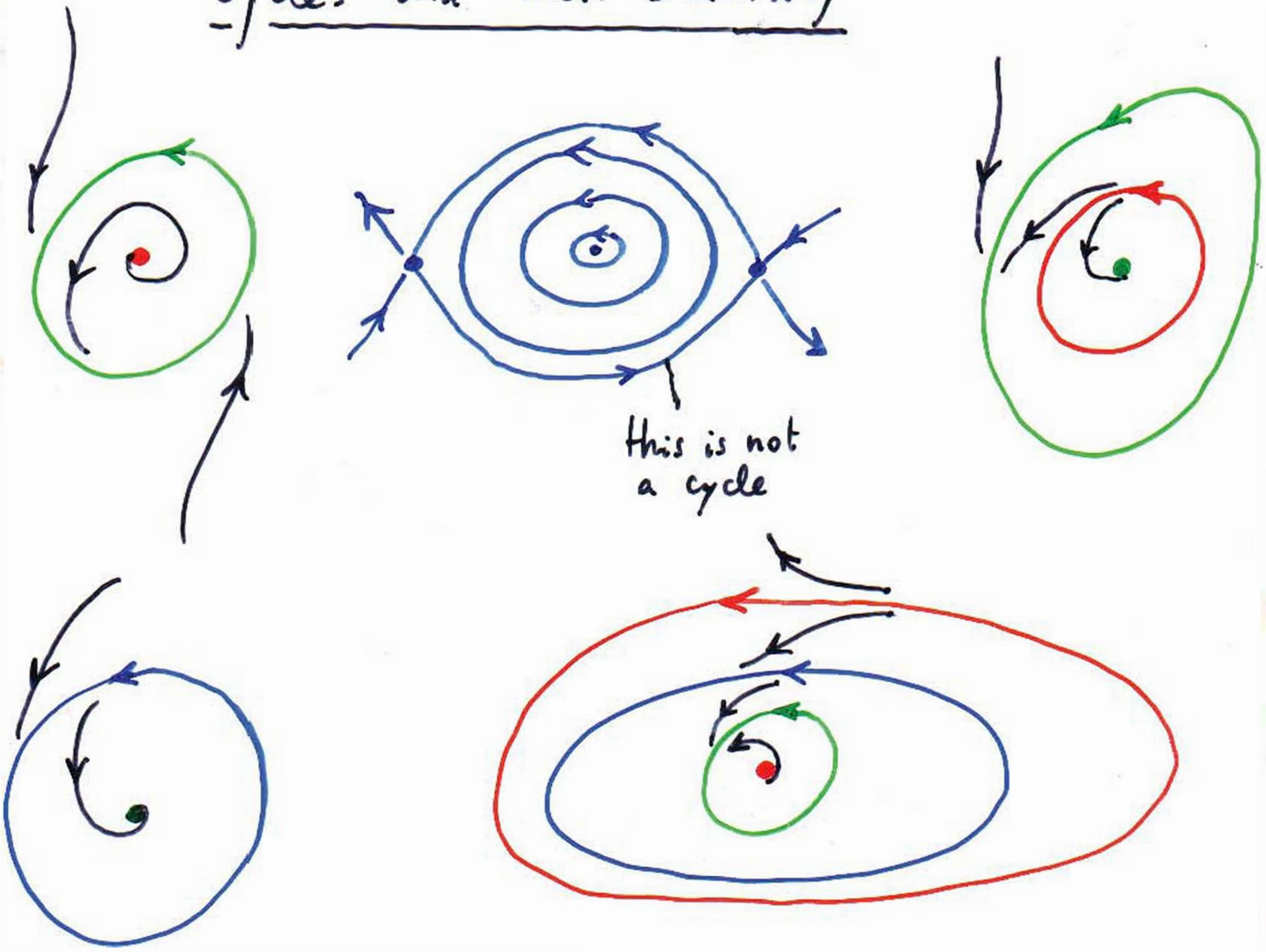
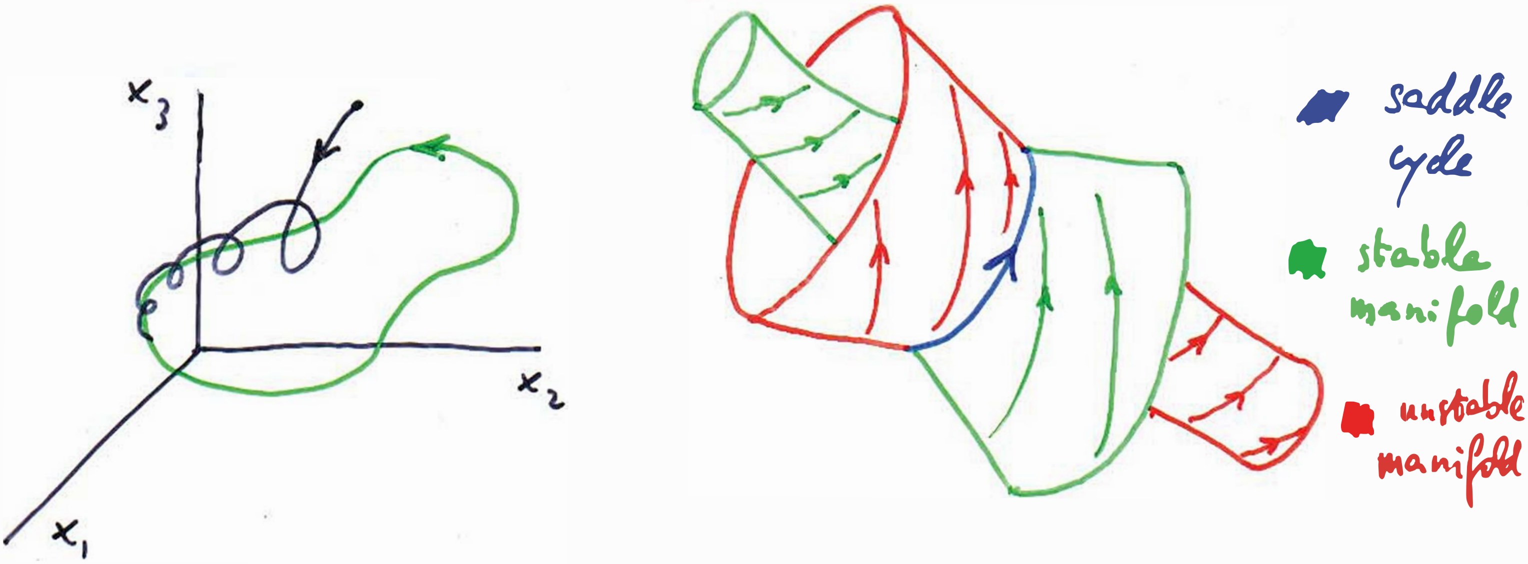
Cycles and their stability



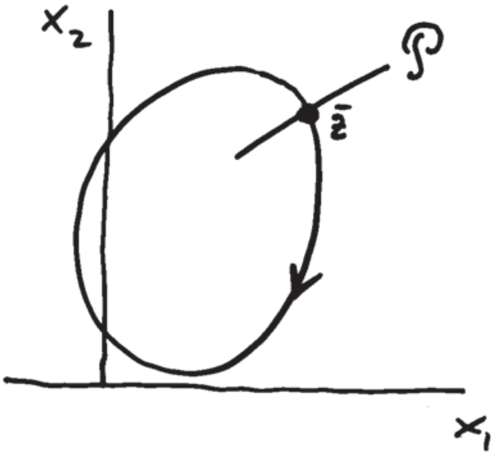


Definition 1 ((asymptotic) stability of a cycle)

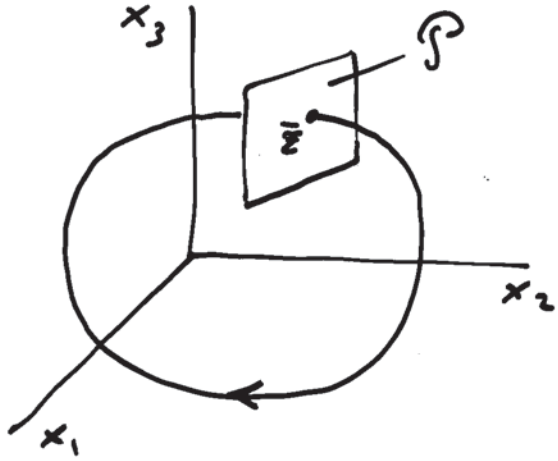
A cycle is (asymptotically) stable if a tube can be constructed around it such that all trajectories starting inside the tube remain in it and tend for $t \to \infty$ to the cycle.

Similarly for stability and instability

Poincaré section and map



$$\dot{x} = f(x)$$



The Poincaré section $\mathcal{P}$ is transversal to the cycle.

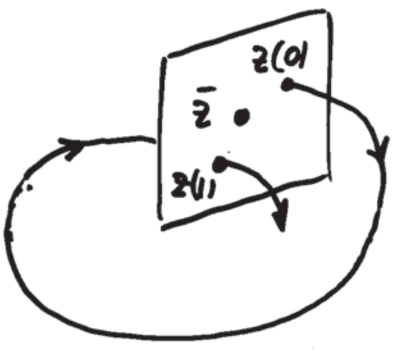
Nearby trajectories (in the tube) cross $\mathcal{P}$.

The cycle intersects $\mathcal{P}$ at $\bar{z}$.

Consider a nearby trajectory starting from a point

$$z(0) \in \mathcal{P}$$

The trajectory will return to $\mathcal{P}$, close to $\bar{z}$, at point $z(1)$ and then at points $z(2)$, $z(3)$, ...



Obviosly the cycle is asymptotically stable if and only if

$$z(t) \longrightarrow \bar{z} \qquad \text{for } t \to \infty$$

for all $z(0)$ close to $\bar{z}$

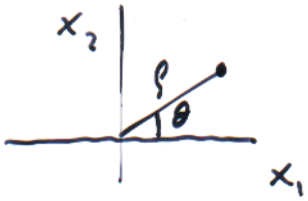
$$z(t+1) = P(z(t)) \quad (*) \qquad \text{with} \qquad \bar{z} = P(\bar{z})$$

Poincaré map (implicitely defined by $\dot{x} = f(x)$)

Theorem 1 The cycle is asymptotically stable if and only if $\bar{z}$ is an asymptotically stable equilibrium of $(*)$

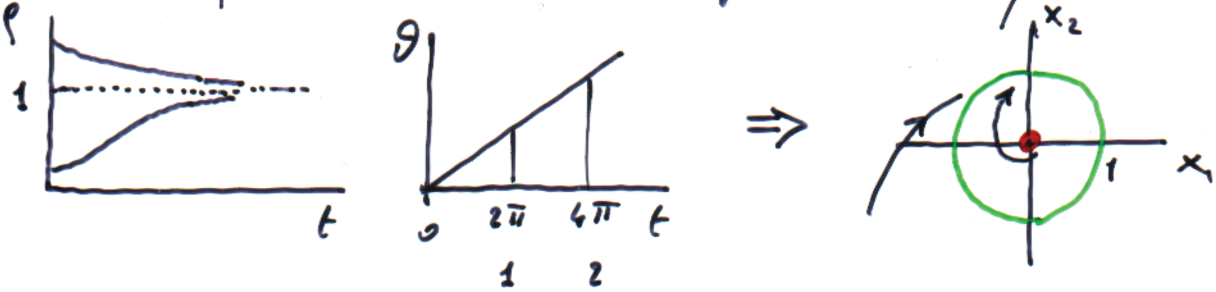
Ex. 1 The simplest limit cycle

$$\begin{cases} \dot{\rho} = \rho(1-\rho) \\ \dot{\vartheta} = 1 \end{cases}$$



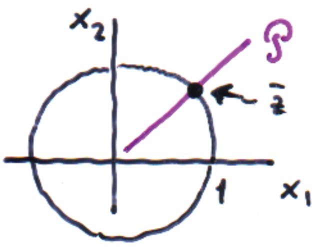
polar coordinates

The two equations are decoupled $\Rightarrow$ easy



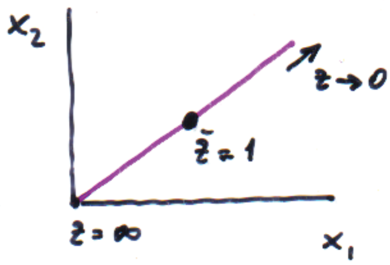
Let us see how we can proceed with Theorem 1.

(i) Fix a Poincaré section $\mathcal{P}$, for example, the straight line $x_2 = x_1$



(ii) Fix a coordinate system z on $\mathcal{P}$, for axample,

$$z = \frac{1}{\rho}$$



(iii) Derive the Poincaré map $z(t+1) = P(z(t))$ t = integer

For this let $y(t) = \frac{1}{\rho(t)}$, t = real. Hence, $\dot{y} = -\frac{1}{\rho^2}\dot{\rho}$, i.e.

$$\dot{y} = -\frac{1}{\rho^2}\rho(1-\rho) = -y+1 \Rightarrow y(t) = e^{-t}y(0) + 1 - e^{-t}$$

Since $y(0) = z(0)$, $y(2\pi) = z(1)$, $y(4\pi) = z(2)$, ...

we obtain

$$z(t+1) = e^{-2\pi} z(t) + 1 - e^{-2\pi} \qquad (*)$$

(iv) The Poincaré map $(*)$ is a first-order discrete-time linear system. Its eigenvalue is $e^{-2\pi} < 1$. Hence $(*)$ is asymptotically stable, i.e. $z(t) \to \bar{z}$

Linearization of the Poincaré map

On the Poincaré section $\mathcal{P}$ we have

$$z(t+1) = P\left(z(t)\right) \qquad (*)$$

with $\bar{z} = P(\bar{z})$.

The stability of $\bar{z}$ in $(*)$ can be studied through linearization

$$\delta z(t) = z(t) - \bar{z} \qquad \delta z(t+1) = \left.\frac{\partial P}{\partial z}\right|_{\bar{z}} \delta z(t)$$

$\left.\frac{\partial P}{\partial z}\right|_{\bar{z}}$ is the Jacobian matrix of the Poincaré map

$\left.\frac{\partial P}{\partial z}\right|_{\bar{z}}$ has $(n-1)$ eigenvalues because z is $(n-1)$-dimens

<u>**Theorem 2**</u> (linearization method + Thm. 1)

Let μ_i, $i = 1, \ldots, n-1$ be the eigenvalues of $\left.\frac{\partial P}{\partial z}\right|_{\bar{z}}$

Then $|\mu_i| < 1 \ \forall i \Rightarrow$ the cycle is asympt. stable

<u>Remark 1</u> The eigenvalues $\mu_1, \ldots, \mu_{n-1}$ are sometimes called <u>Floquet's coefficients</u>

<u>**Theorem 3**</u> If one (or more) $|\mu_i| > 1$ the cycle is unstable

<u>Remark 2</u> If $|\mu_i| \leq 1$ and at least one $|\mu_i| = 1$ nothing can be said on the stability of the cycle

Second order systems

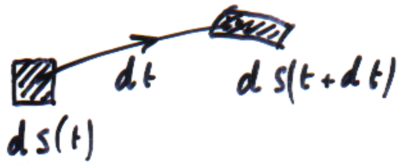
$$\dot{x}_1 = f_1(x_1 x_2)$$
$$\dot{x}_2 = f_2(x_1 x_2)$$

special properties hold for these systems

Theorem 4 (Bendixon) (non existence condition)

If the divergence $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$ does not change sign in a domain $\Omega \subset R^2$ (and, at most, annihilates on a line) then there are no cycles in Ω.

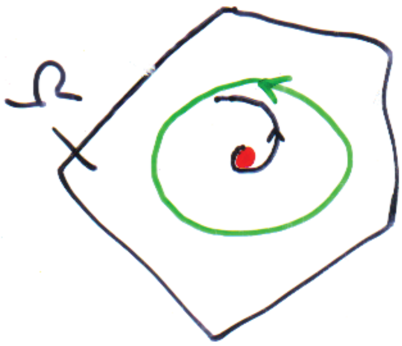
Proof



$$(\dot{ds}) = (\mathrm{div}\, f) \cdot ds$$

Thus, $\mathrm{div}\, f > 0 \Rightarrow$ trajectories diverge

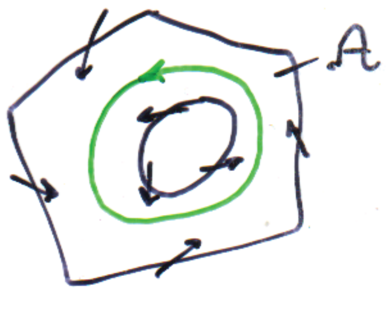
$\mathrm{div}\, f < 0 \Rightarrow$ trajectories converge



If there exists a stable cycle with an unstable equilibrium inside, trajectories must diverge (close to the equilibrium) and converge (close to the limit cycle).

Theorem 5 (Poincare') (existence condition)

If there are no equilibria in an annular domain $A \subset R^2$ and trajectories enter in A from its boundary, then there is at least one cycle in A.



Proof. Since trajectories entering in A cannot cross one each other and cannot tend toward an equilibrium, they must converge to a cycle.

Remark 3 Theorem 5 holds also if trajectories leave A.

Equilibria and cycles

Consider a second-order system

$$\dot{x}_1 = f_1(x_1\ x_2)$$
$$\dot{x}_2 = f_2(x_1\ x_2)$$

and assume that all its equilibria are hyperbolic (i.e. all Jacobian $\left.\frac{\partial f}{\partial x}\right|_{\bar{x}}$ have eigenvalues with non zero real parts)

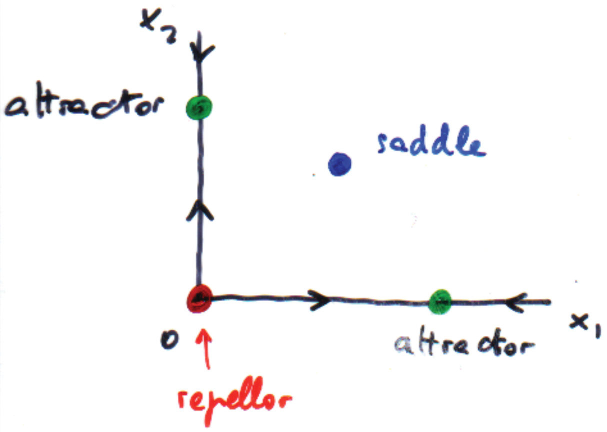
Then, the following property holds.

Theorem 6 (Poincare')

Inside a cycle there must be an odd number of equilibria. More precisely, if s is the number of saddles, there must be $(s+1)$ equilibria which are not saddles.

Ex. 2 Bacteria competition (see Ex. 7 Lecture 3)

$$\begin{cases} \dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ \dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_1 x_2 \end{cases}$$

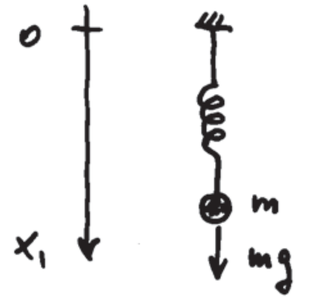


The system is positive (trajectories remain in the 1-st quadrant)

Cycles cannot exist in the first quadrant, because they would have inside either no equilibria or one saddle.

Problems

P.1 (2) - Consider the following mechanical system and assume that the spring is "linear" (i.e. its force is proportional to x_1) and that a viscous friction acts on the mass (i.e. a force opposed to the motion and proportional to the speed x_2 of the mass)



Write the state equations and prove that in such a system cycles cannot exist.

P.2 (1) - Derive the state equations $\dot{x}_1 = \cdots$ $\dot{x}_2 = \cdots$ of the model described in polar coordinates in Ex. 1.

P.3 (1) - Say why cycles cannot exist in the epidemic model

$$\dot{x}_1 = -\alpha x_1 x_2 \qquad x_1 = \text{susceptible}$$
$$\dot{x}_2 = \alpha x_1 x_2 - \beta x_2 \qquad x_2 = \text{infective}$$

P.4 (3) - Consider the prey-predator model described in problem P.1 of Lecture 4, and prove, by means of Theorem 5 (actually a slight modification of Thm. 5), that under conditions (1), (2) there exists a cycle (<u>Hint</u>: define the annular domain $\mathcal{A}$ by using segments of the x_1 and x_2 axis for the external boundary and a small circle around $\bar{x}^{(3)}$ for the internal boundary.