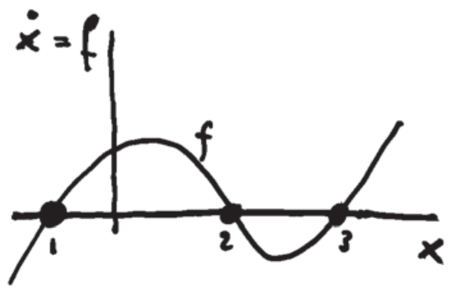
Equilibria and isoclines

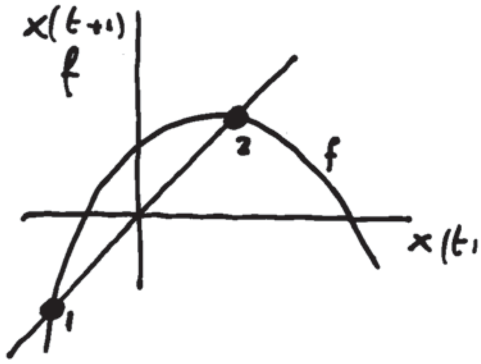
Definition 1 (equilibrium)

An equilibrium is a state $\bar{x}$ such that $x(0)=\bar{x}$ implies $x(t)=\bar{x} \quad \forall t \geq 0$.

Consequence In continuous-time systems $\dot{x}=f(x)$ equilibria $\bar{x}$ are such that $f(\bar{x})=0$, while in discrete-time systems $x(t+1)=f(x(t))$ equilibria $\bar{x}$ are such that $\bar{x}=f(\bar{x})$

Ex. 1 1-st order systems

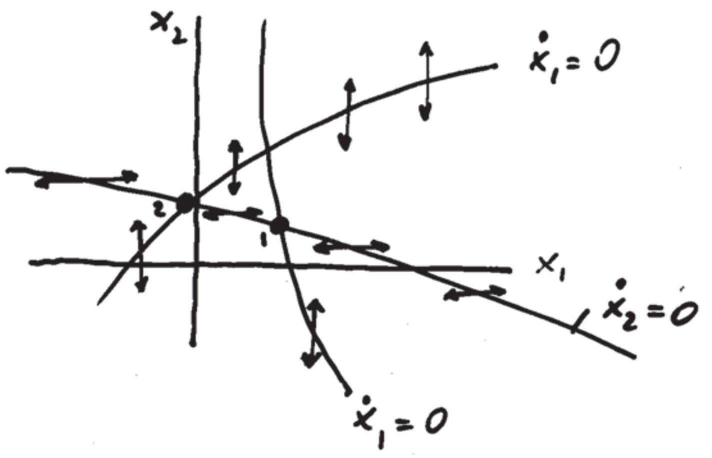


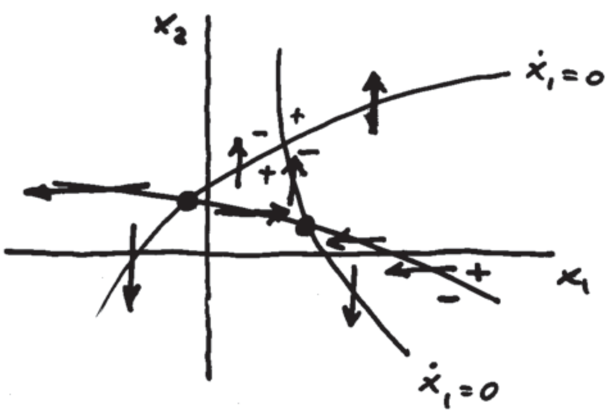


Ex. 2 - 2-nd order continuous time systems

$$\dot{x}_1 = f_1(x_1\ x_2) \quad \Rightarrow \quad \dot{x}_1=0 \quad f_1(x_1\ x_2)=0$$
$$\dot{x}_2 = f_2(x_1\ x_2) \quad \Rightarrow \quad \dot{x}_2=0 \quad f_2(x_1\ x_2)=0$$

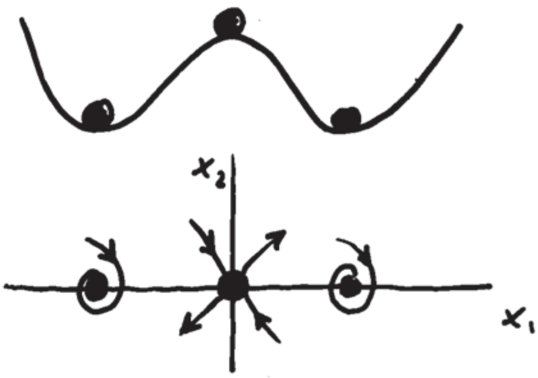
(null) isoclines

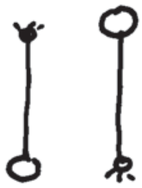


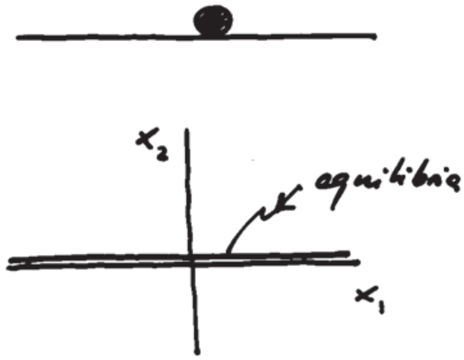


Multiplicity of equilibria

We have already seen that equilibria can be multiple



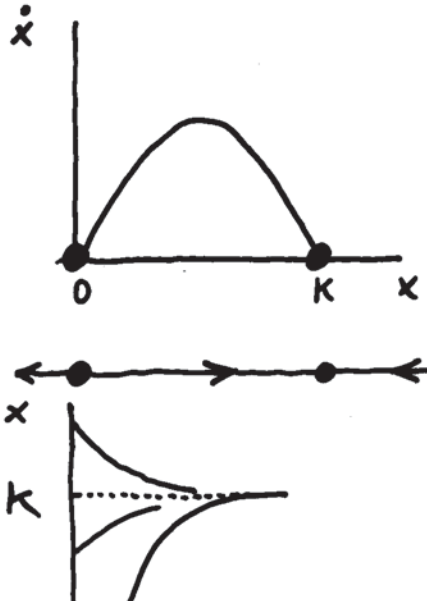




Ex. 3. Logistic growth

$x(t)$ = biomass at time t

$$\left.\begin{aligned} \dot{x} &= n(x)\cdot x - m(x)\,x \\ n(x) &= n_0 - \alpha x \\ m(x) &= m_0 + \beta x \end{aligned}\right\} \Rightarrow \quad \dot{x} = r x \left(1 - \frac{x}{K}\right) \qquad \begin{aligned} r &= n_0 - m_0 \\ K &= \frac{n_0 - m_0}{\alpha + \beta} \end{aligned}$$



two equilibria $\bar{x} = 0$

$\bar{x} = K$ carrying capacity

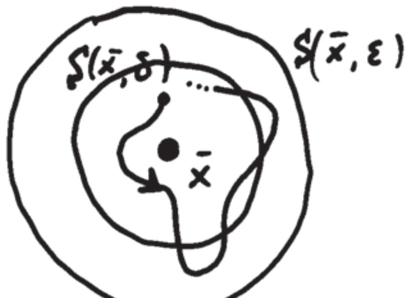
the population biomass tends toward its carrying capacity

Stability

Definition 2 ((local) stability) (Liapunov 1892)

An equilibrium $\bar{x}$ is (locally) stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x(0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| < \varepsilon \qquad \forall x(0),\ t > 0$$



There is no small perturbation of the state, after which the system moves far away from the equilibrium

Definition 3 (asymptotic stability)

A stable equilibrium $\bar{x}$ is asymptotically stable if all perturbed trajectories $x(t)$ tend to $\bar{x}$ for $t \to \infty$

In words: any small perturbation is asymptotically absorbed

Definition 4 ((local) instability)

An equilibrium which is not stable is called unstable.

Definition 5 (basin of attraction)

Given an asymptotically stable equilibrium $\bar{x}$ the set

$$B(\bar{x}) = \{ x(0) : x(t) \to \bar{x} \}$$

is called basin of attraction

Definition 6 (global stability)

If $B(\bar{x})$ coincides with R^n (with the exception of a set of zero measure) $\bar{x}$ is called globally stable

Examples

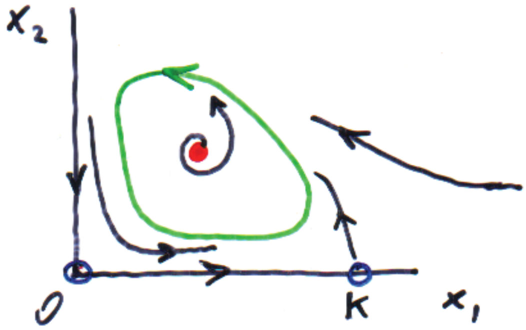
Ex. 4 Logistic growth

0 is unstable and K is stable

K is asymptotically stable

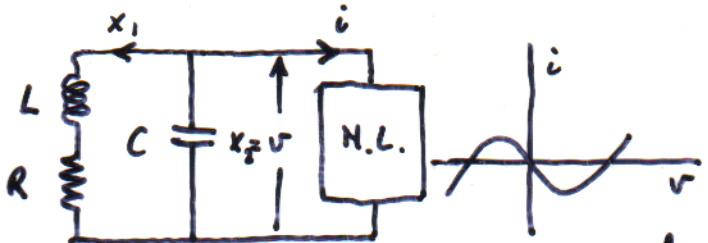
K is globally stable (in R_+)

Ex. 5 Prey-predator model

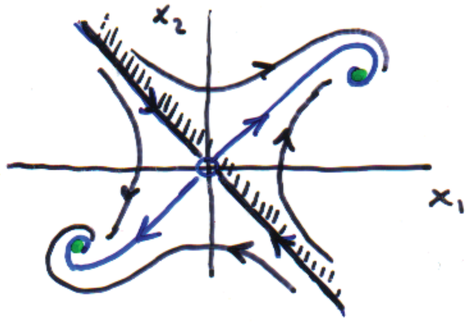


There are three equilibria. They are all unstable (one is a focus and two are saddles)

Ex. 6 Electric circuit (bistable)



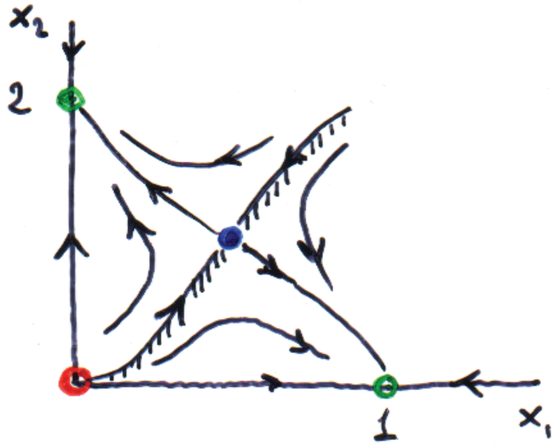
$$i = -gv + kv^3$$



Ex. 7 Bacteria competition

$$\dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2$$

$$\dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_1 x_2$$



4 equilibria : 2 stable, 2 unstable

control from 2 to 1 : antibiotic + yeast

Problems

P. 1.[1] Prove that a linear system $\dot{x} = Ax$ has a unique equilibrium (namely, the origin $\bar{x} = 0$) if A is non singular (i.e. if A does not have a null eigenvalue). What happens if A is singular? Formulate the analogous results for discrete-time linear systems.

P. 2 -[1] Sketch the state portrait of the following mechanical system and indicate the basin of attraction of the two stable equilibria

P. 3 -[1] Consider the following discrete-time 1-st order model

$$x(t+1) = \frac{1}{2 - x(t)}$$

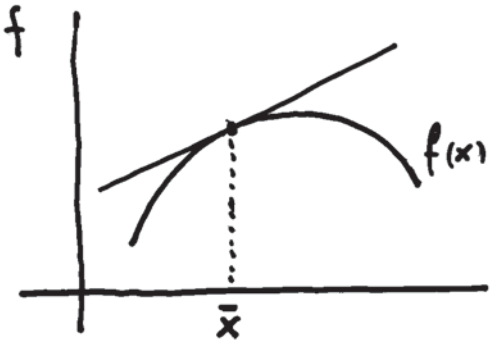
and show that this model has only one equilibrium, namely $\bar{x} = 1$. Then, use Moran's construction to prove that the equilibrium is unstable. Finally, observe that all trajectories tend toward $\bar{x}$, even if $\bar{x}$ is unstable. (This model describes the evolution of a genetic desease in a population: t is the generation and $x(t)$ is the probability that a randomly selected individual of generation t is sick).

P. 4 -[2] Using the model described in Ex. 7 and the isoclines try to give a formal support to the state portrait shown in Ex. 7.

Local approximation

$$y = f(x) \qquad y = f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} (x - \bar{x}) + \frac{1}{2} \left.\frac{d^2 f}{dx^2}\right|_{\bar{x}} (x - \bar{x})^2 + \cdots$$

local approximation $\qquad y = f(\bar{x}) + \left.\dfrac{df}{dx}\right|_{\bar{x}} (x - \bar{x})$



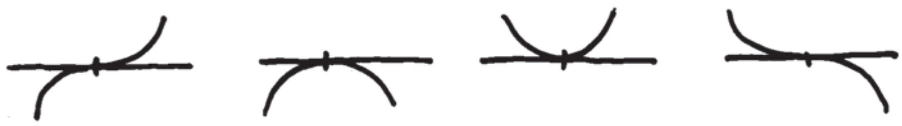
geometric interpretation of linearization

The sign of $\left.\dfrac{df}{dx}\right|_{\bar{x}}$ is important

$\left.\dfrac{df}{dx}\right|_{\bar{x}} > 0 \qquad f(\cdot)$ is locally increasing

$\left.\dfrac{df}{dx}\right|_{\bar{x}} < 0 \qquad f(\cdot)$ is locally decreasing

$\left.\dfrac{df}{dx}\right|_{\bar{x}} = 0 \qquad$ critical case : nothing can be said



Jacobian matrix (1)

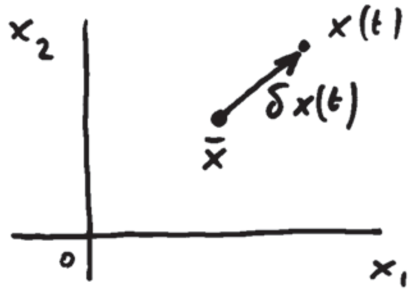
$$\dot{x}(t) = f(x(t))$$

$$\bar{x} = \text{equilibrium} \ (f(\bar{x}) = 0)$$

Taylor's expansion

$$\dot{x} = f(\bar{x}) + \left[\frac{\partial f}{\partial x}\right]_{\bar{x}} (x - \bar{x}) + \underbrace{\cdots}_{O(x-\bar{x})^2}$$

$$\left[\frac{\partial f}{\partial x}\right] = \underline{\text{Jacobian}}\,(\text{matrix}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = J$$



$$\delta x(t) = x(t) - \bar{x}$$

$$\delta\dot{x}(t) = (\dot{x}(t) =) \left[\frac{\partial f}{\partial x}\right]_{\bar{x}} \delta x(t) + \cdots$$

Linearized system

$$\boxed{\delta\dot{x} = J\,\delta x}$$

Ex. 1 - Logistic growth

$\dot{x} = r x \left(1 - \frac{x}{K}\right)$ $\qquad \bar{x} = 0 \Rightarrow \delta\dot{x} = r x$

$\left(J = r - \frac{2 r x}{K}\right)$ $\qquad \bar{x} = K \Rightarrow \delta\dot{x} = -r x$

Jacobian matrix (2)

$$x(t+1) = f(x(t))$$

$$\bar{x} = \text{equilibrium} \quad (\bar{x} = f(\bar{x}))$$

$$x(t+1) = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} (x(t) - \bar{x}) + \cdots$$

$$x(t+1) = \bar{x} + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} (x(t) - \bar{x}) + \cdots$$

$$\delta x(t+1) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} \delta x(t) + \cdots \qquad \boxed{\delta x(t+1) = J\, \delta x(t)}$$

linearized system

Ex. 2 - Age structured population

$x_i(t)$ = # individuals of age i in year t

$$x_1(t+1) = x_3(t) \cdot F(x_3(t)) \qquad \leftarrow F = \text{fertility}$$

$$x_2(t+1) = s_2 x_1(t) \qquad s = \text{survival}$$

$$x_3(t+1) = s_3 x_2(t) + s^* x_3(t)$$

mature individuals

$$\delta x(t+1) = J\, \delta x(t)$$

$$J = \begin{vmatrix} 0 & 0 & \overbrace{F(\bar{x}_3)}^{(1-s^*)/s_2 s_3} + \left.\frac{\partial F}{\partial x_3}\right|_{\bar{x}_3} \bar{x}_3 \\ s_2 & 0 & 0 \\ 0 & s_3 & s^* \end{vmatrix}$$

Linearisation method

$$\left.\begin{array}{l}\dot{x} = f(x) \\ x(t+1) = f(x(t))\end{array}\right\} \bar{x} = \text{equilibrium} \;\Rightarrow\; J = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}}$$

Theorem 1 J = asympt. stable $\Rightarrow$ $\bar{x}$ = asympt. stable

In words: if the linearised system is asymptotically stable, the equilibrium $\bar{x}$ is such

Theorem 2 J = exponentially unstable $\Rightarrow$ $\bar{x}$ = unstable

at least one eigenvalue of J is strictly unstable ($Re(\lambda_i) > 0$ or $|\lambda_i| > 1$)

Comment If J is simply stable or weakly unstable nothing can be said on the stability of $\bar{x}$.

Ex. 3. Logistic growth

$$\dot{x} = r x \left(1 - \frac{x}{K}\right)$$

$\bar{x} = 0$ $J = r \Rightarrow \bar{x}$ = unstable

$\bar{x} = K$ $J = -r \Rightarrow \bar{x}$ = asympt. stable

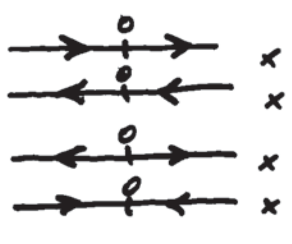
Ex. 4 - Quadratic and cubic systems

$\dot{x} = x^2$ $\bar{x} = 0$ $J = 2x|_{\bar{x}} = 0 \Rightarrow ?$

$\dot{x} = -x^2$ $\bar{x} = 0$ $J = -2x|_{\bar{x}} = 0 \Rightarrow ?$

$\dot{x} = x^3$ $\bar{x} = 0$ $J = 0 \Rightarrow ?$

$\dot{x} = -x^3$ $\bar{x} = 0$ $J = 0 \Rightarrow ?$



Problems

P. 1.[3] Consider the prey - predator model

$$\begin{cases} \dot{x}_1 = r\, x_1 \left(1 - \dfrac{x_1}{K}\right) - \dfrac{a\, x_1}{b + x_1}\, x_2 & \text{prey} \\[2ex] \dot{x}_2 = e\, \dfrac{a\, x_1}{b + x_1}\, x_2 - d\, x_2 & \text{predator} \end{cases}$$

and assume that the positive parameters a, b, e, d, r, K satisfy the following inequalities

$$b < K \qquad (1) \qquad\qquad \frac{ea}{d} > \frac{K+b}{K-b} \qquad (2)$$

. Show that under conditions (1, 2) the system has two trivial equilibria

$$\bar{x}^{(1)} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad \text{and} \qquad \bar{x}^{(2)} = \begin{vmatrix} K \\ 0 \end{vmatrix}$$

and one strictly positive equilibrium

$$\bar{x}^{(3)} = \begin{vmatrix} \bar{x}_1 \\ \bar{x}_2 \end{vmatrix} \qquad \bar{x}_i > 0 \quad i = 1, 2$$

- Show, through linearization, that $\bar{x}^{(1)}$, $\bar{x}^{(2)}$ and $\bar{x}^{(3)}$ are unstable.
- Compare with Ex. 5 of Lecture 3.

P. 2.[3] Discuss, through linearization, the stability of the four equilibria shown in Ex. 7 of Lecture 3 (pay attention to inequalities among parameters).

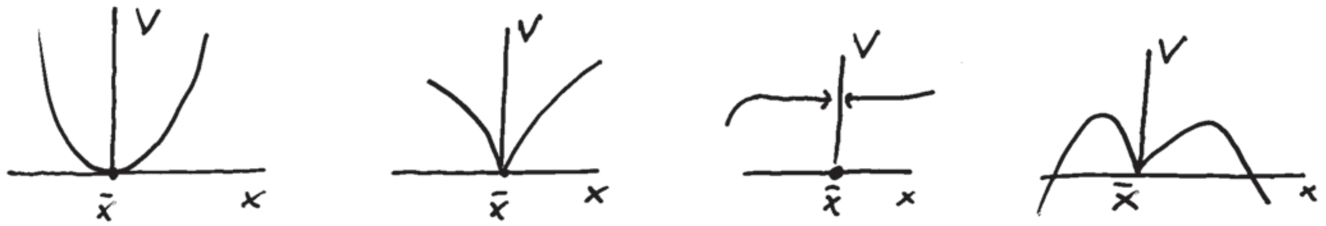
POSITIVE DEFINITE FUNCTIONS

$x \in \mathbb{R}^n \qquad V(x) : \mathbb{R}^n \to R$

__Definition__ $V(\cdot)$ is positive definite at $\bar{x}$ iff

(a) $V(\bar{x}) = 0$

(b) $V(x) > 0 \quad \forall x \neq \bar{x}$ in a neighborhood of $\bar{x}$



In $\mathbb{R}^2$

$V(x) = x_1^2 + x_2^2 \qquad V(x) = x_1 - \log x_1 + x_2 - \log x_2 - 2$

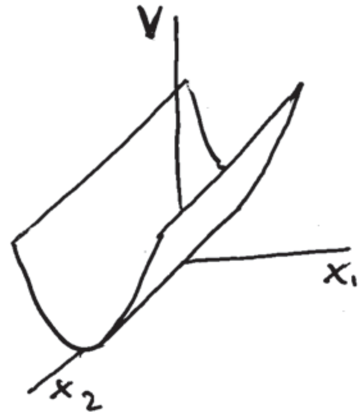
$V(x) = a x_1^2 + 2 b x_1 x_2 + c x_2^2$ with $a > 0$, $ac > b^2$

__Definition__ $V(\cdot)$ is positive semidefinite at $\bar{x}$ iff

(a) $V(\bar{x}) = 0$

(b) $V(x) \geq 0 \quad \forall x \neq \bar{x}$ in a neighborhood of $\bar{x}$

(c) $V(x) = 0$ for some x close to $\bar{x}$



$V = x_1^2$ is posit. semid. in $\mathbb{R}^2$

__Definition__ $V(\cdot)$ is negative (semi)definite iff $-V(\cdot)$ is positive (semi)definite

LIAPUNOV METHOD [1892]

$$\dot{x} = f(x) \qquad f(\bar{x}) = 0$$

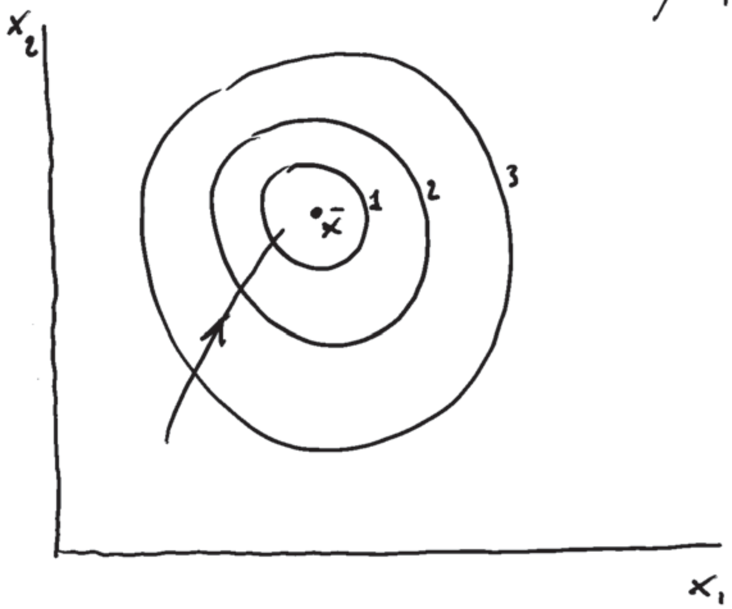
(a) $V(\cdot)$ posit. def. at $\bar{x}$

(b) $V(\cdot)$ regular (continuous with continuous derivatives)

(c) $\dot{V}(x) = \frac{\partial V}{\partial x} f = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \dots + \frac{\partial V}{\partial x_n} f_n$ negative definite at $\bar{x}$

$$\Downarrow$$

$$\bar{x} = \text{asympt. stable}$$



(a) + (b) ⇒ the lines at $V = \text{const.}$ are closed and ordered close to $\bar{x}$.

$\dot{V}$ neg. def. ⇒ trajectories cross the lines $V = \text{const.}$ transversally from outside toward inside

<u>Remark</u> A function $V(\cdot)$ satisfying conditions (a), (b) and (c) is called a Liapunov function

<u>Krasowskii's method</u>

(a) + (b) + $\dot{V}$ negative semidefinite + Krasowskii condition (*)

$$\Downarrow$$

$$\bar{x} = \text{asympt. stable}$$

$$K = \{ x : \dot{V}(x) = 0 \}$$

(*) K does not contain trajectories close to $\bar{x}$ but $\neq \bar{x}$

LA SALLE METHOD

$\dot{x} = f(x) \qquad f(\bar{x}) = 0$

$V(\cdot)$ is a Liapunov (or Krasowskii) function

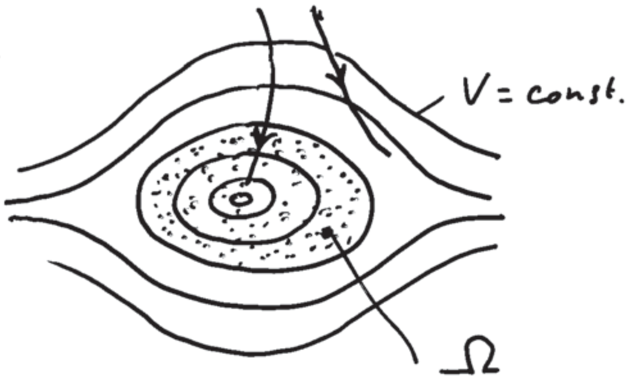
$$\Downarrow$$

$$\bar{x} = \text{asympt. stable}$$

$$\Downarrow$$

$$B(\bar{x}) = \,?$$

La Salle : $B(\bar{x}) \supset \Omega$ = the largest region contained in a closed line $V(x) = \text{const.}$ in which Liapunov (or Krasows.) conditions hold



Remark Ω is an estimate (from below) of $B(\bar{x})$

Remark quadratic positive definite functions $V(\cdot)$ have always closed lines $V = \text{const.}$

EXAMPLE

$$\begin{cases} \dot{x}_1 = \alpha\left(-x_1 + \beta^2 x_1^3 - x_2\right) \\ \dot{x}_2 = \gamma\, x_1 \end{cases}$$

(Van der Pol equation)

$\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow \bar{x} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$ is the only equilibrium

$$J = \begin{vmatrix} -\alpha & -\alpha \\ \gamma & 0 \end{vmatrix}$$

$\mathrm{tr}\, J = -\alpha < 0 \qquad \det J = \alpha\gamma > 0$

$\Downarrow$

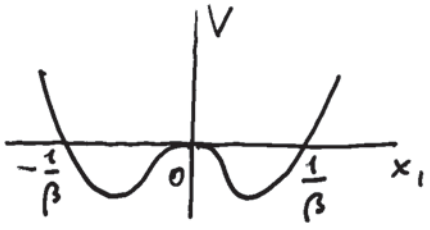
λ_1 and λ_2 have neg. real part

$\Downarrow$

$\bar{x}$ = asympt. stable

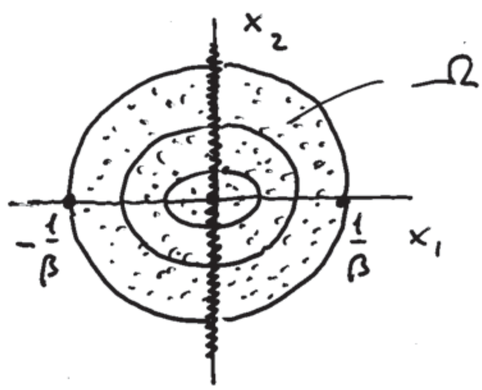
$$V(x) = \frac{1}{2}\frac{1}{\alpha} x_1^2 + \frac{1}{2}\frac{1}{\gamma} x_2^2$$

$$\dot{V} = \frac{1}{2}\frac{1}{\alpha} 2x_1 \alpha\left(-x_1 + \beta^2 x_1^3 - x_2\right) + \frac{1}{2}\frac{1}{\gamma} 2x_2 \gamma x_1 = -x_1^2\left(1 - \beta^2 x_1^2\right)$$



$\dot{V}$ is negative semidefinite at $\bar{x} = 0$

Since V is quadratic the lines V = const. are closed



In Ω Krasowskii conditions are satisfied because $\dot{x}_1 = -\alpha x_2$ on the x_2 axis