Dynamical Systems

discrete-time

$t =$ integer

$x(t) =$ state at time t

$x(t) \in R^n$ $\qquad$ $n =$ order

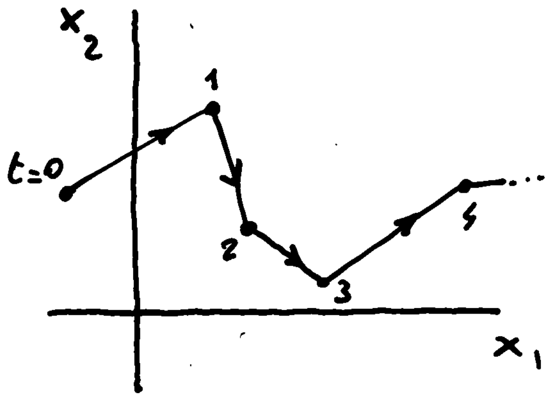
continuous-time

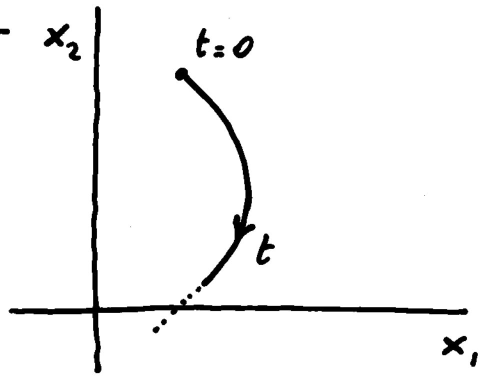
$t =$ real

$x(t) =$ state at time t

$x(t) \in R^n$

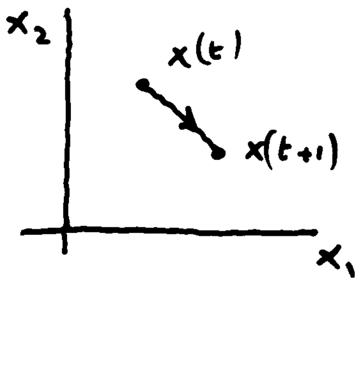
trajectory



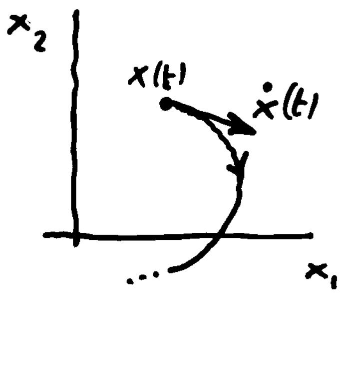


state equations

$$x(t+1) = f(x(t))$$

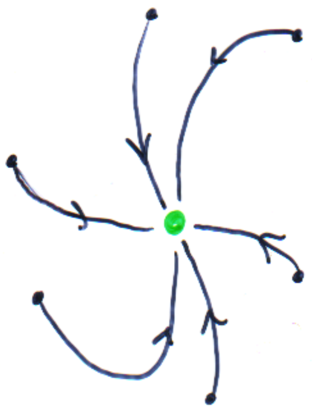


$$\dot{x}(t) = f(x(t))$$



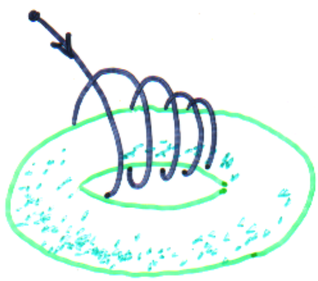
Naïve view at attractors

EQUILIBRIUM ($n \geq 1$)

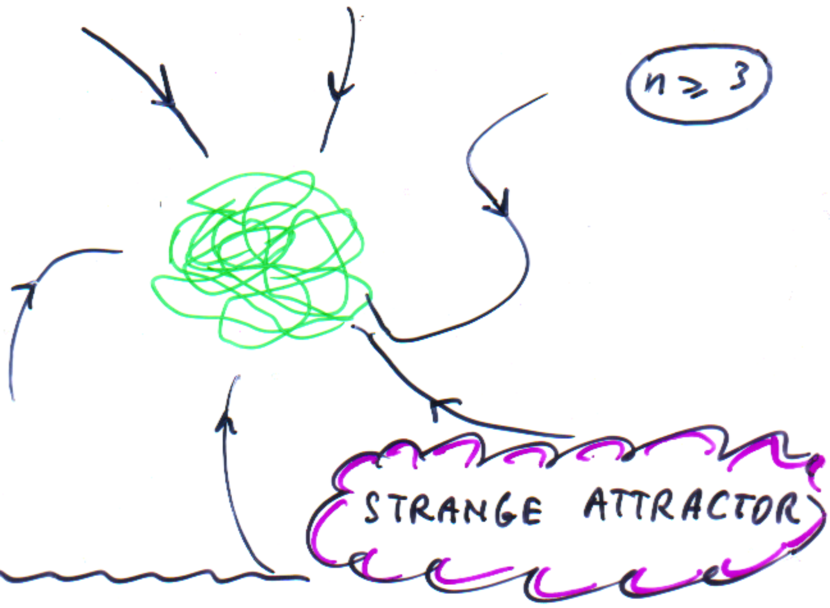


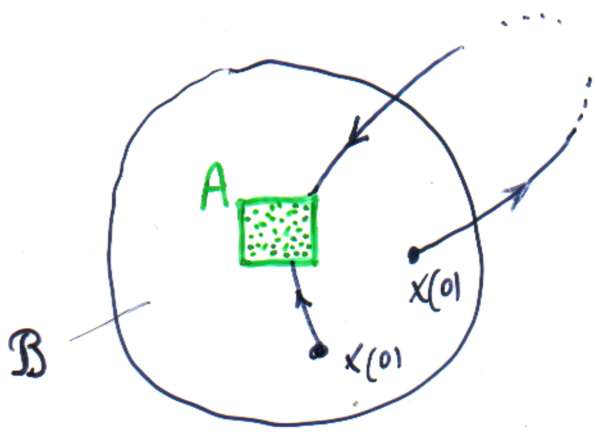
CYCLE ($n \geq 2$)

TORUS ($n \geq 3$)

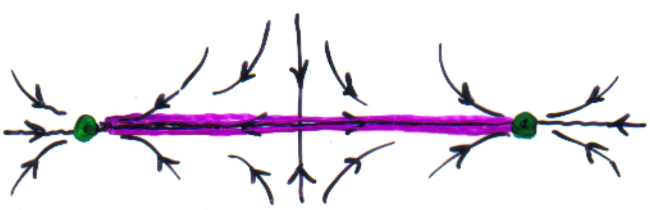


($n \geq 3$)





A
- invariant set $x(0) \in A \Rightarrow x(t) \in A \quad \forall t$
- attracts an open set $\mathcal{B}$ of initial conditions
- minimal



 is not minimal

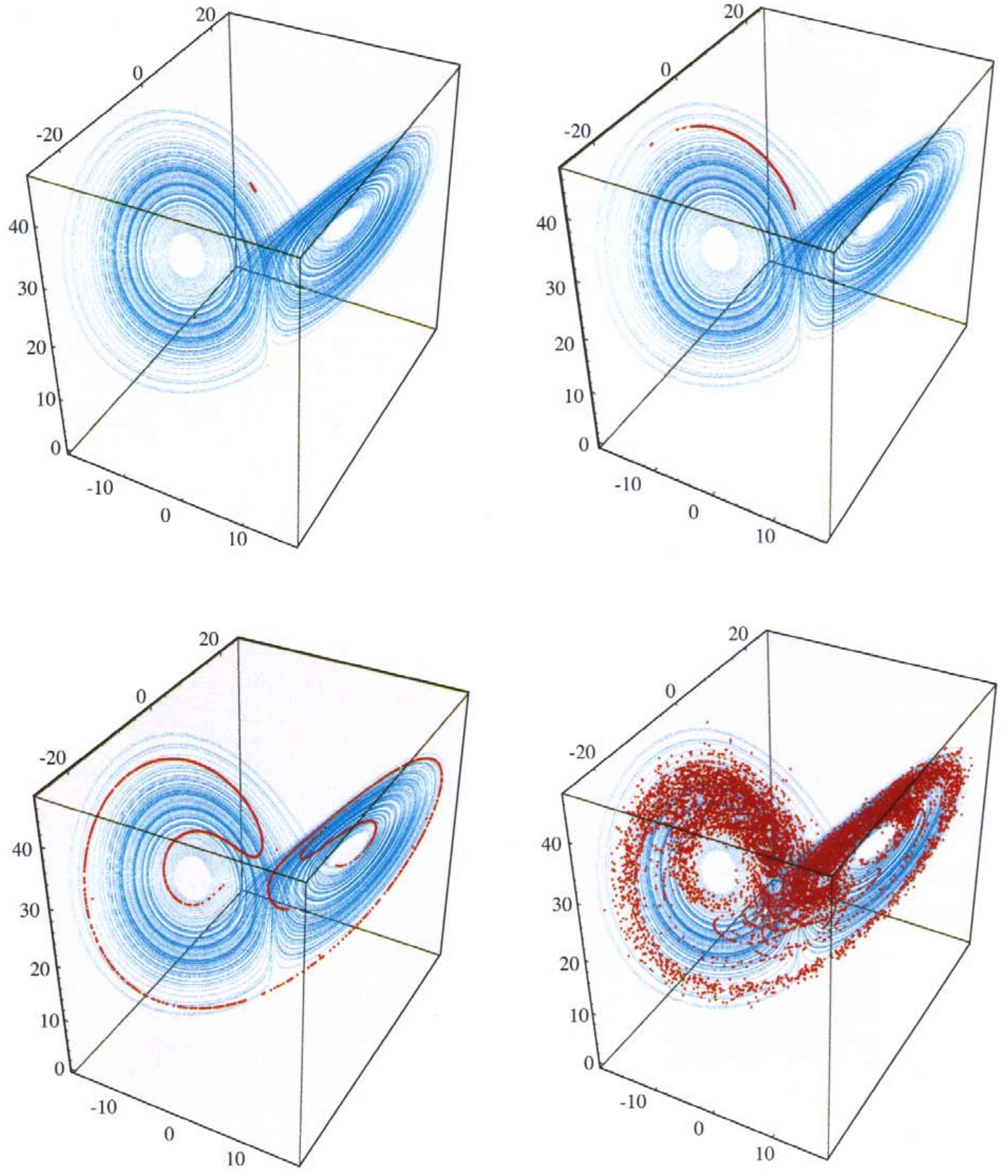
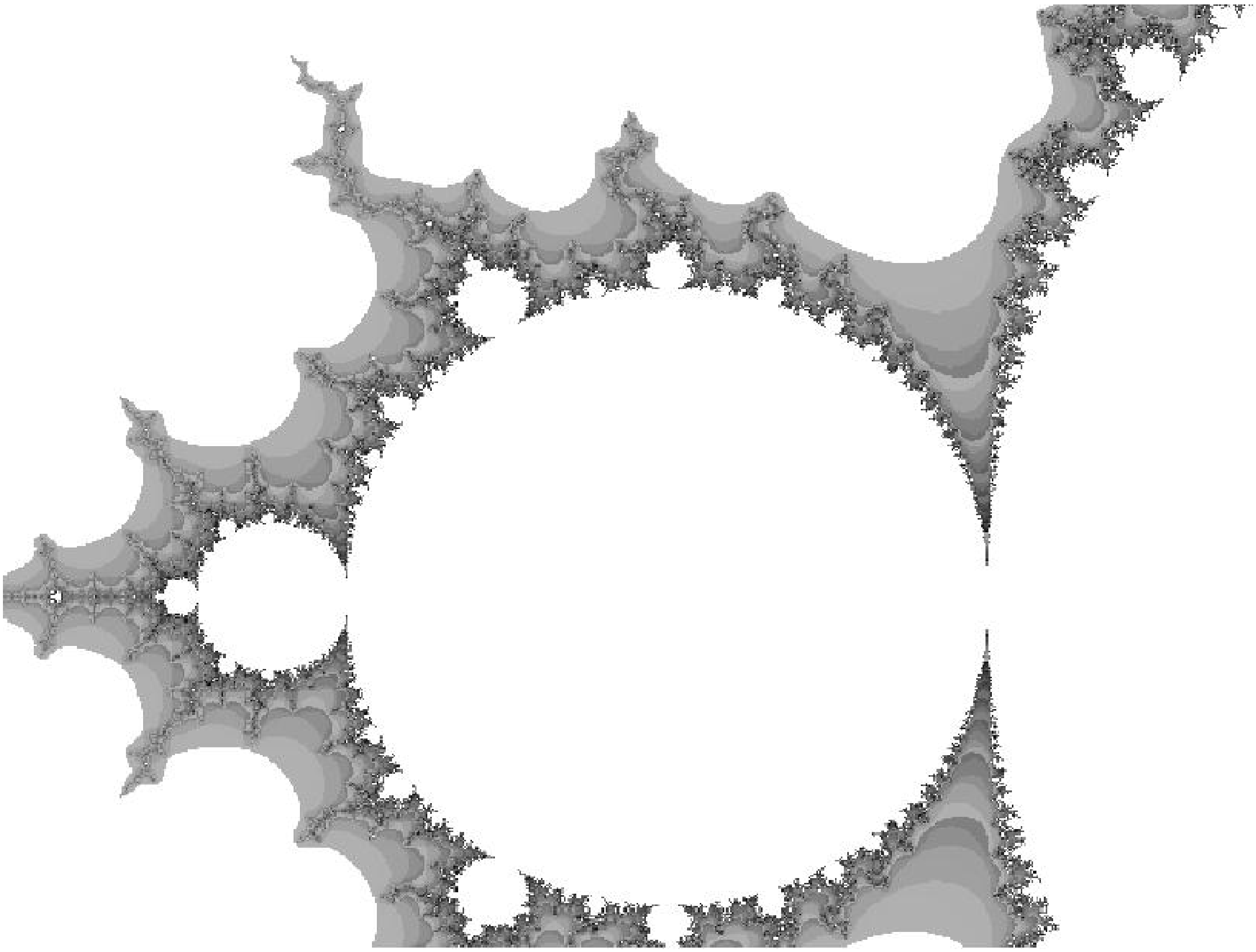
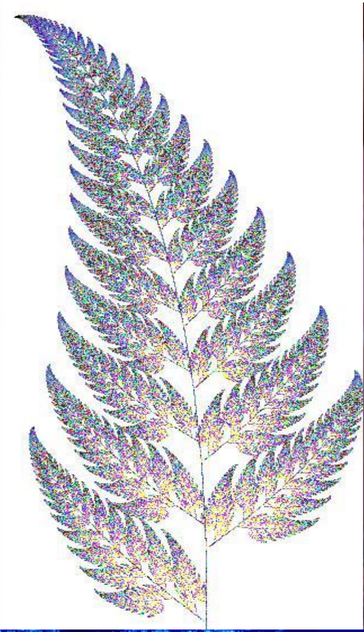
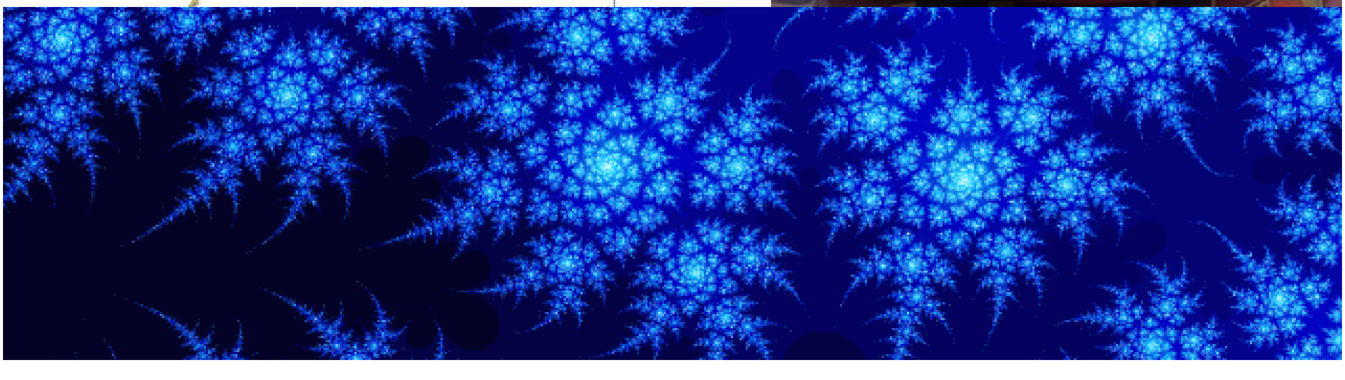


Plate 2: Divergence of nearby trajectories on the Lorenz attractor (Section 9.3). The Lorenz attractor is shown in blue. The red points show the evolution of a small blob of 10,000 nearby initial conditions, at times $t = 3$, 6, 9, and 15. As each point moves according to the Lorenz equations, the blob is stretched into a long thin filament, which then wraps around the attractor. Ultimately the points spread over much of the attractor, showing that the final state could be almost anywhere, even though the initial conditions were almost identical. This sensitive dependence on initial conditions is the signature of a chaotic system.

Plate inspired by a similar illustration in Crutchfield et al. (1986). Numerical integration and computer graphics by Thanos Siapas, using Equation (9.2.1) with parameters $\sigma = 10$, $b = 8/3$, $r = 28$.





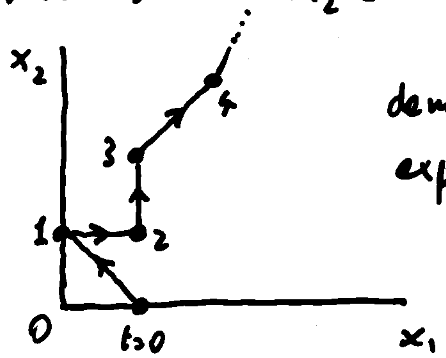


Examples

Ex. 1 Fibonacci's rabbits (13-th century)

t = generation $\quad x_1$ = young rabbits $\quad x_2$ = adult rabbits

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = x_1(t) + x_2(t) \end{cases} \qquad x(0) = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$



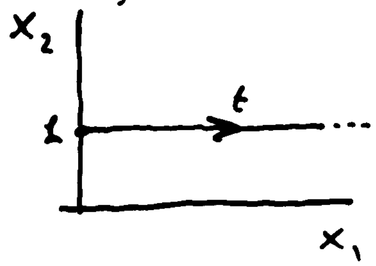
demographic explosion

Ex. 2 Newton's law (17-th century)

$u = m \times acc.$

x_1 = position $\quad x_2$ = velocity $\quad u$ = force $\equiv 0$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{cases}$$



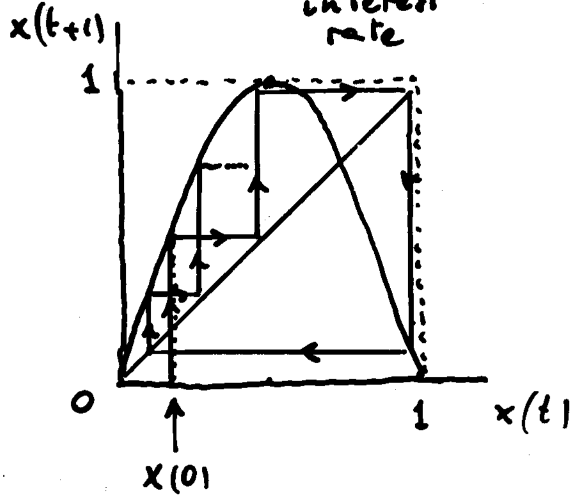
$$x(0) = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

Ex. 3 quadratic map (May - Guckenheimer)

t = year $\quad x(t)$ = capital (\$)

$$x(t+1) = (1+i)\,x(t) - \tau\left[(1+i)\,x(t)\right]^2 = r\,x(t)\left(1 - \frac{x(t)}{k}\right)$$

i ↑ interest rate $\quad \tau$ ↑ tax



$r = 4 \quad k = 1$

Moran's construction

$x(0) \longrightarrow x(1) \longrightarrow x(2) \longrightarrow x(3) \longrightarrow \cdots$

Classes of dynamical systems

Reversible systems

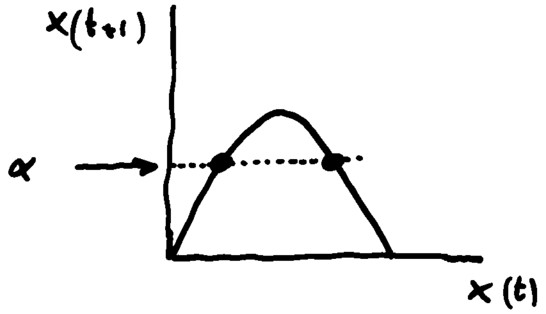
$x(t) \rightarrow x(t-\tau)$

ODE's can be integrated backward in time $\Rightarrow$

$\Rightarrow$ continuous-time systems are reversible

By contrast, discrete-time systems can be irreversible

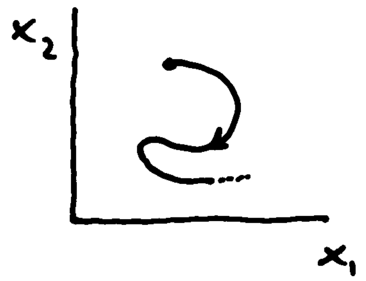
Ex. 4 quadratic map.



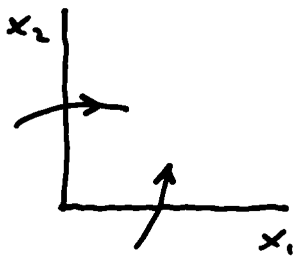
there are two possible predecessors of the state α

Positive systems

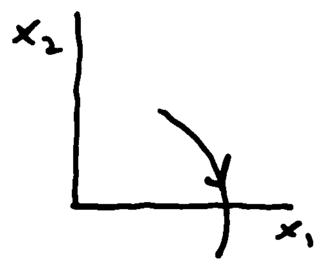
$x(t) \geq 0 \Rightarrow x(t+\tau) \geq 0$



R_+^n is an invariant set



this is possible



this is impossible

Linear systems

$x(t+1) = A\,x(t)$ $\qquad\qquad$ $\dot{x}(t) = A\,x(t)$

$A = n \times n$ matrix

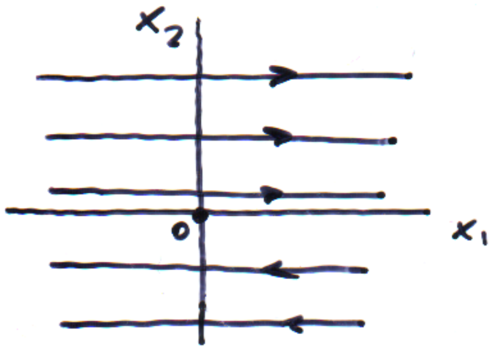
Ex. 5. Fibonacci's rabbits $\qquad A = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

Ex. 6. Newton's law $\qquad A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

State portrait

Definition: the state portrait is a set of trajectories in state space from which one can understand the behaviour of the system for any $x(0)$.

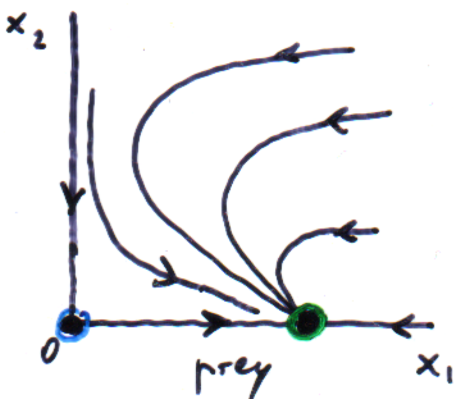
Ex. 7 Newton's law



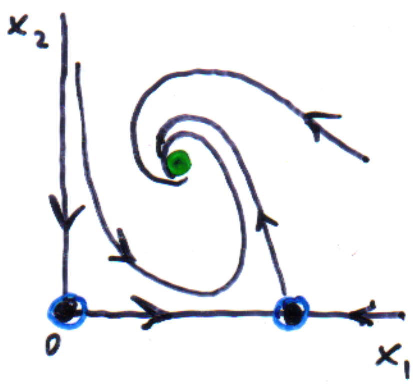
the velocity remains constant

the mass goes to infinity for $t \to \infty$

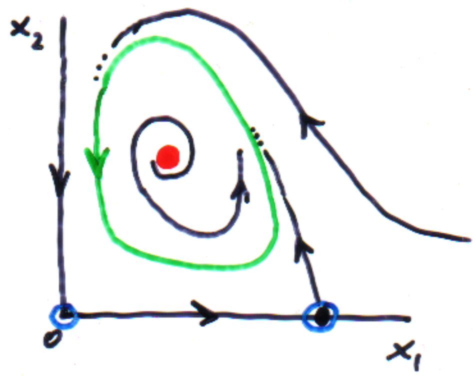
Ex. 8 Prey-predator behaviour



predator extinction

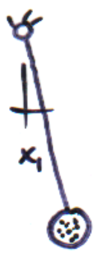


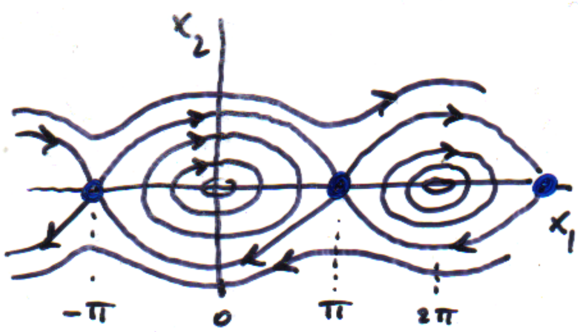
steady coexistence



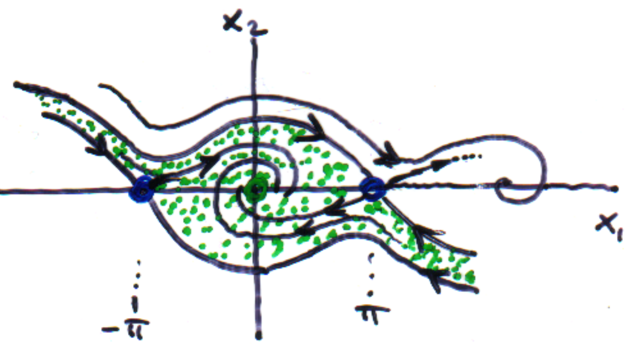
cyclic coexistence

Ex. 9 Pendulum





without friction



with friction

Problems

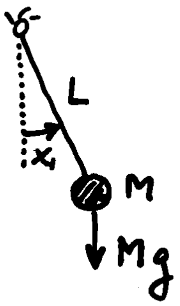
P. 1[(2)]. Consider the second-order continuous-time dynamical system (Kermack - McKendrick (1927))

$$\begin{cases} \dot{x}_1 = -\alpha\, x_1 x_2 \\ \dot{x}_2 = \alpha\, x_1 x_2 - \beta x_2 \end{cases}$$

with α and β positive constant.

- Prove that the system is positive.
- Interpret x_1 and x_2 as densities of susceptible and infected individuals.
- Express your intuitive ideas on the dynamics of the epidemics through a state portrait.
- Simulate the system for $\alpha = \beta = 1$ and find out if your intuition was correct.

P. 2[(2)]. Consider the following pendulum



x_1 = angle
x_2 = angular velocity

and assume that there is a viscous friction, i.e. a momentum $\mathcal{M} = -\alpha x_2$ opposed to the motion and proportional to the angular velocity x_2.

- Write the state equations of the pendulum
- Determine the state portrait through simulation for $M = L = 1$ and $\alpha = 0.1$.