Analyzing planar systems with pplane

1 Diseases spread control

On an island with $N = 1000$ inhabitants, a disease spreads out. The disease has the following characteristics:

1. "Infected" people don't realize to be infected during an incubation period of $T = 10$ days, on average, since in this period they are not affected by any symptom.
2. "Susceptible" people are infected by random encounters with infected people.
3. At the end of the incubation period, infected people show an hepatic problem, and so they go to the doctor/hospital, that report to the authorities the fact. At the same time they recover and become "immune", and do not infect susceptible people anymore.

- Show that the disease spread can be modeled by the following equations

$$\begin{aligned}\dot{S}(t) &= -\alpha S(t)I(t)\\ \dot{I}(t) &= \alpha S(t)I(t) - \beta I(t)\\ d(t) &= \beta I(t)\end{aligned}$$

  where $S(t)$, and $I(t)$, are the number of susceptible, and infected people at time t respectively, $d(t)$ is the number of reports obtained by the authorities at time t, and αdt and β are the probability to be infected in a contact and the recovering rate.

- Show that the system is positive, i.e. $(S(0), I(0)) \geq 0 \Rightarrow (S(t), I(t)) \geq 0$.
- Locate the equilibria of the system, and discuss their stability through linearization.
- Discuss the existence of limit cycles.
- Sketch a possible state portrait using the null-cline method.
- Verify the obtained results using `pplane`.

- Knowing that the number of reports obtained by the authorities at different time t are

t	0	7	10	11	12
$d(t)$	1	13	30	35	40

  estimate the parameter values of α and β.

- Give an interpretation of the obtained results, and propose a control action to reduce the disease spread.

2 Prey-predator model

Given the Rosenzweig-MacArthur model

$$\begin{aligned} \dot{x}_1 &= rx_1\left(1 - \frac{x_1}{K}\right) - \frac{ax_1}{b+x_1}x_2 \\ \dot{x}_2 &= e\frac{ax_1}{b+x_1}x_2 - mx_2 \end{aligned}$$

where x_1 and x_2 are the prey and the predator densities, $r = m = 1$, $a = 6$, $b = 2$, $e = 1/2$ and $K \in \{0.5, 2, 5\}$.

- Show that the model is positive, i.e. $(x_1(0), x_2(0)) \geq 0 \Rightarrow (x_1(t), x_2(t)) \geq 0$
- Analyze the model dynamic in absence of predators ($x_2 = 0$) and in absence of preys ($x_1 = 0$).
- Locate the equilibria of the system, and discuss their stability through linearization.
- Let $K = 5$, and sketch the trajectories of the system in the neighborhood of the equilibria.
- Discuss the existence of limit cycles.
- Sketch a possible full state portrait.
- Verify the obtained results using `pplane`.